Synopsis and Contents

of the thesis entitled as

Generalization of Equations, Solutions, and Techniques for KdV-type Evolution Equations

to be submitted to the University of Delhi
for the award of the degree of

Doctor of Philosophy
In
Mathematics

By

BRIJ MOHAN

under the supervision of

Prof. Raj Kumar

Department of Mathematics
Faculty of Mathematical Sciences
University of Delhi, Delhi-110007, India
September 2025

Synopsis

To understand the complex physical systems in oceanography, fluid dynamics, plasma physics, optics, and engineering, the nonlinear partial differential equations (PDEs) [1–5] play a significant role. They have different wave structures, including solitons [6–9], kink waves [10, 11], and rogue waves [12–18], which play a significant role due to their stability, localized energy, and sudden emergence, respectively. We investigate these nonlinear evolution equations mathematically for a theoretical understanding of integrable systems as well as for analyzing real-world physical processes. The exact solutions of nonlinear PDEs are investigated using powerful symbolic techniques such as the Hirota bilinear method [8], the symbolic bilinear technique [9], and the direct symbolic approach [20].
The integrability test for the novel equations is necessary to obtain the N-soliton solution with the symbolic techniques. We analyze the integrability of the evolution equations with Painlevé analysis [5, 20], satisfying the Painlevé property for the observed solutions. Due to the complexity of these symbolic techniques, we utilize computer algebra systems, such as *Mathematic*a and *Maple*, to analyze and observe these techniques. We explore the graphical analysis of the obtained solutions for the studied nonlinear evolution equations.
This thesis develops and applies novel analytical and symbolic techniques [8,9,13,20] to investigate integrable higher-dimensional nonlinear KdV-type evolution equations. It spans kink-solitons, generalized soliton solutions, higher-order rogue waves, and dispersive solitons with new insights into their dynamics, parameter dependencies, and graphical representations. We extend the direct method given by Hirota [8] to a symbolic bilinear technique [9], which broadens the truthfulness of exact solutions to generalized solutions with arbitrary parameters. Across the chapters, multiple new integrable equations are proposed, tested for Painlevé integrability, and transformed into bilinear or trilinear forms [15, 20] for the construction of solutions.
In this research, we formulate novel integrable equations [5, 11, 19] and investigate these equations by new techniques [9, 20]. We propose a direct symbolic approach and a symbolic bilinear technique to explore rogue waves and generalized soliton solutions, respectively. With the usefulness of the Cole-Hopf transformation [5, 9, 11], we formulate the bilinear and trilinear forms that enable the systematic construction of solitons, kink waves, dispersive solitons, and rogue waves. Furthermore, the proposed novel symbolic bilinear technique generalizes Hirota's bilinear method by incorporating arbitrary parameters. This research contributes to extending the expoloration of integrable models in soliton theory. Also, it establishes new methodological pathways for generating exact solutions of higher-dimensional nonlinear PDEs.

The thesis is organized into six main chapters. **Chapter 1** serves as an **introduction** and lays the foundation and motivation of the work with a detailed literature survey. It explains the significance and contributions of this study and discusses the novelty of the thesis. This chapter provides general descriptions for the newly proposed techniques and existed approaches in a step-by-step explanation. It begins with an explanation of the preliminary steps of Hirota bilinear approach covering Cole-Hopf transformation and associated studies, and gives detailed explanations of newly proposed methods Symbolic bilinear technique (SBT) [9] and Direct symbolic approach (DSA) [20]. The chapter also provides a concise review of the mathematical preliminaries of tranformations, integrability, exact solutions such as solitons, kink-solitons and rogue waves supported by relevant literature.

Chapter 2, titled as "**Kink-solitons and Rogue waves of a new integrable (3+1)-dimensional generalized KdV-type equation** [11]," proposes a novel (3+1)-dimensional Painlevé integrable KdV-type equation that generalizes well-known equations in soliton theory and nonlinear sciences. It illustrates the Painlevé analysis to establish the complete integrability of the proposed equation. It employs the Cole-Hopf transformations to get the bilinear equation in an auxiliary function and further construct it into Hirota's bilinear form. Utilizing the Hirota bilinear technique, it obtains the soliton solutions of kink types and their interactions up to the third order. It examines the rogue waves of the higher order using a direct symbolic approach up to the third order. For constructing the rogue waves, it transforms the investigated equation from (3+1)-dimensional to a (1+1)-dimensional partial differential equation and form its Hirota bilinear form in transformed variables. This chapter demonstrates the dynamics for the obtained kink-soliton and rogue wave solutions with appropriate parameter values using the symbolic system *Mathematica*. The interaction solutions of rogue waves show the dominating nature of more giant waves over smaller waves. It analyzes the rogue wave graphics in both the transformed and original variables.

In **Chapter 3**, titled as "**Generalized solitary wave solutions using a novel symbolic bilinear approach for nonlinear PDEs**, [9]" we analyze the new generalized soliton solutions for the nonlinear partial differential equations with a novel symbolic bilinear technique. The proposed approach constructs the soliton solutions depending on the arbitrary parameters, which generalizes the soliton solutions with these additional parameters. Examining phase shifts and their dependence on the parameters influences how solitons collide, merge, or pass through each other, which is essential for the nonlinear analysis of solitons. Using the proposed technique, it examines the well-known (1+1)-dimensional Korteweg-de Vries (KdV) [8] and (2+1)-dimensional Kadomtsev-Petviashvili (KP) [9] equations with a comparative analysis of soliton solutions in the Hirota technique. It constructs the generalized solitons solutions for both examined equations up to the third order, providing a better understanding of formed solitons with arbitrary parameter choices. The Cole-Hopf transformations are used to construct the bilinear form in the auxiliary function using Hirota's D-operators for both investigated KdV and KP equations. It discusses the phase shift depending on parameters and compares it to the phase shift in Hirota's soliton solutions. This chapter utilizes *Maple*, a computer algebra system, to obtain the generalized solitons and analyzes the behavior of the obtained solutions by finding the values for the parameters and the relationships among them.

Chapter 4, titled as "**A novel Painlevé-type (3+1)-dimensional evolution equation: Higher-order rogue waves and solitons**, [19]" proposes a new Painlevé integrable generalized (3+1)-dimensional evolution equation. It demonstrates the Painlevé test that claims the integrability of the proposed equation and employs Cole-Hopf transformations to generate the trilinear equation in an auxiliary function that governs the higher-order rogue wave and dispersive-soliton solutions via the symbolic computation approach and dispersive-soliton assumption, respectively. Center-controlled parameters in rogue waves show the different graphical structures with several other parameters. It obtains the solutions for rogue waves up to third-order using direct symbolic approach with appropriate center parameters and other parameters using a generalized procedure for rogue waves. It assumes the dispersive-soliton solution, inspired by Hirota's direct techniques to create dispersive-soliton solutions up to the third order. By applying the symbolic software *Mathematica*, we demonstrate the graphical analysis for rogue waves with diverse center parameters and dispersive solitons using dispersion relation to showcase the interaction behavior of the solitons.

In **Chapter 5**, titled as "**Analysis of Cole-Hopf transformations, solitons, and rogue waves for a (2+1)-dimensional shallow water wave equation**, [17]" we investigate a (2+1)-dimensional shallow water wave equation of ion-acoustic waves in plasma physics. It comprehensively analyzes Cole-Hopf transformations concerning dimensions x, y, and t and obtains the dispersion for a phase variable of this equation. It shows that the soliton solutions are independent of the different logarithmic transformations for the investigated equation, and explores the linear equations in the auxiliary function f present in Cole-Hopf transformations. This chapter studies the equation's first- and second-order rogue waves using a generalized N-rogue wave expression from the N-soliton Hirota technique. We generate the rogue waves by applying a direct symbolic approach with β and γ as center parameters. It creates rogue wave solutions for first- and second-order using direct computation for appropriate choices of several constants in the equation and center parameters. It obtain a trilinear equation by transforming variables ξ and y via logarithmic transformation for u in the function F. Also, It demonstrates the graphics of the soliton and center-controlled rogue wave solutions with suitable choices of parameters.

Chapter 6, titled as "**Rogue waves for a generalized (3+1)-dimens-ional KdV-type nonlinear wave equation**, [18]" explores the behavior of higher-order rogue waves within a (3+1)-dimensional generalized nonlinear wave equation in liquid-containing gas bubbles. It creates the investigated equation's Hirota D-operator bilinear form, and employs a generalized formula with real parameters to obtain the rogue waves up to the third order using the direct symbolic approach. The analysis reveals that the second and third-order rogue solutions produce two and three-waves, respectively. To gain deeper insights, it uses the Cole-Hopf transformation on the transformed variables ξ and η to produce a bilinear equation. Using system software *Mathematica*, the graphical analysis presents the dynamics for the obtained solutions in transformed ξ, η and original spatial-temporal coordinates x, y, z, t. These visualizations reveal rogue waves' intricate structure and evolution, capturing their localized interactions and significant presence within nonlinear systems. It demonstrates that rogue waves, characterized by their substantial height and sudden appearance, are prevalent in various nonlinear events. The equation examined in this study offers valuable insights into the evolution of longer waves with smaller amplitudes, which is particularly relevant in fields such as fluid dynamics, dispersive media, and plasma.

At the end of the thesis, we conclude our research work with remarks and future scope.

− − − − − − − − ∗ ∗ ∗ − − − − − − − −

References

[1] Adem A.R., Podile T.J., Muatjetjeja B.A.: *A Generalized (3+1)-Dimensional Nonlinear Wave Equation in Liquid with Gas Bubbles: Symmetry Reductions; Exact Solutions; Conservation Laws. Int. J. Appl. Comput. Math* **9**:82 (2023)

[2] Xing Lü, Si-Jia Chen: *N-soliton solutions and associated integrability for a novel (2+1)-dimensional generalized KdV equation*, Chaos Soli Fract. **169**, 113291, (2023)

[3] Mohan B., Kumar S.: *Painlevé analysis, restricted bright-dark N-solitons, and N-rogue waves of a (4+1)-dimensional variable-coefficient generalized KP equation in nonlinear sciences*, Nonlinear Dynamics, **112**:11373-11382 (2024)

[4] Wazwaz A.M., Hammad M.A., El-Tantawy S.A.: *Bright and dark optical solitons for (3 + 1)-dimensional hyperbolic nonlinear Schrödinger equation using a variety of distinct schemes*, Optik, **270**:170043, (2022)

[5] Kumar S., Mohan B.: *A generalized nonlinear fifth-order KdV-type equation with multiple soliton solutions: Painleve analysis and Hirota Bilinear technique*, Physica Scripta, 97(12):125214, (2022)

[6] He C., Tang, Y., Ma, W. et al.: *Interaction phenomena between a lump and other multi-solitons for the (2+1)-dimensional BLMP and Ito equations*, Nonlinear Dyn. **95**:29–42 (2019)

[7] Ma W.X.: *N-soliton solutions and the Hirota conditions in (2+1)-dimensions*, Opt Quant. Electron, **52**:511, (2020)

[8] Hirota R.: *The direct method in soliton theory.* Cambridge: Cambridge University Press, (2004).

[9] Mohan B., Kumar S.: *Generalization and analytic exploration of soliton solutions for nonlinear evolution equations via a novel symbolic approach in fluids and nonlinear sciences*, Chinese Journal of Physics, **92**:10-21, (2024)

[10] Yang L., Gao B.: *Solitons, kink-solitons and breather solutions of the two-coupled incoherent nonlinear Schrödinger equation*, Nonlinear Dyn **112**:5621-5633 (2024)

[11] Mohan B., Kumar S., Kumar R.: *On investigation of kink-solitons and rogue waves to a new integrable (3+1)-dimensional KdV-type generalized equation in nonlinear sciences*, Nonlinear Dyn, **113**:10261-10276, (2025)

[12] Yang X., Zhang Z., Wazwaz A.M., Wang, Z.: *A direct method for generating rogue wave solutions to the (3+1)-dimensional Korteweg-de Vries Benjamin-Bona-Mahony equation*, Physics Letters A, **449**, 128355, (2022)

[13] Kumar S., Mohan B., Kumar R.: *Newly formed center-controlled rogue wave and lump solutions of a generalized (3+1)-dimensional KdV-BBM equation via symbolic computation approach*, Physica Scripta, **98**(8):085237, (2023)

[14] Liu W., Zhang Y.: *Multiple rogue wave solutions for a (3+1)-dimensional Hirota bilinear equation.* Appl. Math. Lett. **98**:184–190 (2019)

[15] Kumar S., Mohan B.: *Bilinearization and new center-controlled N-rogue solutions to a (3+1)-dimensional generalized KdV-type equation in plasmas via direct symbolic approach*, Nonlinear Dynamics, **112**:11373-11382 (2024)

[16] Wang M., Tian B., Sun Y., Zhang Z.: *Lump, mixed lump-stripe and rogue wave-stripe solutions of a (3+1)-dimensional nonlinear wave equation for a liquid with gas bubbles,* Comp. Math. Appl., **79** 576-587 (2020)

[17] Kumar S., Mohan B.: *A novel analysis of Cole-Hopf transformations in different dimensions, solitons, and rogue waves for a (2+1)-dimensional shallow water wave equation of ion-acoustic waves in plasmas*, Physics of Fluids, **35**:127128, (2023)

[18] Mohan B., Kumar S.: *Rogue-wave structures for a generalized (3+1)-dimensional nonlinear wave equation in liquid with gas bubbles*, Physica Scripta, **99**:105291, (2024)

[19] Mohan B., Kumar S., Kumar R.: *Higher-order rogue waves and dispersive solitons of a novel P-type (3+1)-D evolution equation in soliton theory and nonlinear waves*, Nonlinear Dynamics, **111**:20275–20288, (2023)

[20] Kumar S., Mohan B.: *A direct symbolic computation of center-controlled rogue waves to a new Painlevé-integrable (3+1)-D generalized nonlinear evolution equation in plasmas.* Nonlinear Dyn, **111**:16395–16405, (2023)

List of Publications

Thesis is based on the following published research articles:

1. **Brij Mohan**, Sachin Kumar and Raj Kumar: *On investigation of kink-solitons and rogue waves to a new integrable (3+1)-dimensional KdV-type generalized equation in nonlinear sciences*, Nonlinear Dynamics, **113**:10261-10276 (2024), DOI: 10.1007/s11071-024-10792-8.
   Indexing: SCIE, Scopus, MCQ: 0.81 (2023), IF: 5.6, SJR: 1.201 (Q1)

2. **Brij Mohan**, Sachin Kumar: *Generalization and analytic exploration of soliton solutions for nonlinear evolution equations via a novel symbolic approach in fluids and nonlinear sciences*, Chinese Journal of Physics, **92**:10-21 (2024), DOI: 10.1016/j.cjph.2024.09.004.
   Indexing: SCIE, Scopus, MCQ: 0.27 (2024), IF: 4.6, SJR: 0.587 (Q2)

3. **Brij Mohan**, Sachin Kumar and Raj Kumar: *Higher-order rogue waves and dispersive solitons of a novel P-type (3+1)-D evolution equation in soliton theory and nonlinear waves*, Nonlinear Dynamics, **111**:20275-20288 (2023), DOI: 10.1007/s11071-023-08938-1.
   Indexing: SCIE, Scopus, MCQ: 0.81 (2023), IF: 5.6, SJR: 1.201 (Q1)

4. Sachin Kumar, **Brij Mohan**: *A novel analysis of Cole-Hopf transformations in different dimensions, solitons, and rogue waves for a (2+1)-dimensional shallow water wave equation of ion-acoustic waves in plasmas*, Physics of Fluids, **35**:127128 (2023), DOI: 10.1063/5.0185772.
   Indexing: SCIE, Scopus, MCQ: 0.21 (2011), IF: 4.6, SJR: 0.900 (Q1)

5. **Brij Mohan**, Sachin Kumar: *Rogue-wave structures for a generalized (3+1)-dimensional nonlinear wave equation in liquid with gas bubbles*, Physica Scripta, **99**:105291 (2024), DOI: 10.1088/1402-4896/ad7cd9.
   Indexing: SCIE, Scopus, MCQ: 0.08 (2013), IF: 2.6, SJR: 0.388 (Q2)

AMS MSC 2020 Classification and Keywords (of the thesis)

AMS MSC 2020 Classification:

- 35A25 - Other special methods applied to PDEs
- 35C08 – Soliton solutions
- 35C05 – Solutions in closed form
- 35D35 - Generalized solutions to partial differential equations
- 35G20 – Nonlinear higher-order PDEs
- 37J35 - Completely integrable systems, integrability tests
- 35Q51 - Soliton equations
- 35Q53 - KdV equations (Korteweg-de Vries equations)
- 47A07 - Forms (bilinear, multilinear)
- 47J35 – Nonlinear evolution equations
- 68W30 - Symbolic computation and algebraic computation

Keywords:

Nonlinear Partial Differential Equations, Evolution equations, Symbolic Techniques, Hirota Bilinear Method, Direct Symbolic Approach, Symbolic Bilinear Technique, New Evolution Equations, Novel Techniques, Painlevé Analysis, Complete Integrability, Integrable Equations, Hirota Bilinear Form, Soliton, Kink solitons, Rogue Waves, KdV-type Equations, Sallow Water Wave Equation, KP Equation, P-type Evolution Equations.

Table Of Contents