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Difference between Diff and diff & Int and int:

restart;

Diff(u(x, y), x, y) = *diff*(u(x, y), x, y)

$$\frac{\partial^2}{\partial x \partial y} u(x, y) = \frac{\partial^2}{\partial x \partial y} u(x, y) \quad \textbf{(2.1)}$$

Int(sin(x), x) = *int*(sin(x), x)

$$\int \sin(x)\, dx = -\cos(x) \quad \textbf{(2.2)}$$

$INT := Int\left(x \tanh(3\,x), x = 0\,..\frac{Pi}{2}\right) = int\left(x \tanh(3\,x), x = 0\,..\frac{Pi}{2}\right)$

$$INT := \int_0^{\frac{\pi}{2}} x \tanh(3\,x)\, dx = -\frac{13\,\pi^2}{108} + \frac{\pi \ln\left(1 + e^{3\,\pi}\right)}{6} + \frac{\mathrm{dilog}\left(1 + e^{3\,\pi}\right)}{18} \quad \textbf{(2.3)}$$

Value := *evalf*(*rhs*(*INT*))

$$Value := 1.188054672 \quad \textbf{(2.4)}$$

$assume(a > 0, b > 0, a > b)$:

$INT2 := Int\left(\frac{1}{a + b\cos(\mathrm{theta})}, \mathrm{theta} = 0\,..Pi\right) = int\left(\frac{1}{a + b\cos(\mathrm{theta})}, \mathrm{theta} = 0\,..Pi\right);$

$$INT2 := \int_0^{\pi} \frac{1}{a\sim + b\sim\cos(\theta)}\, d\theta = \frac{\pi}{\sqrt{a\sim^2 - b\sim^2}} \quad \textbf{(2.5)}$$

Expand and Combine:

expand(tan(x + y));

$$\frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)} \quad \textbf{(3.1)}$$

$combine\left(2\cos\left(\frac{(x+y)}{2}\right)\cos\left(\frac{(x-y)}{2}\right)\right);$

$$\cos(y) + \cos(x) \tag{3.2}$$

combine$\left(7\sin(x) - 56\sin(x)^3 + 112\sin(x)^5 - 64\sin(x)^7\right)$

$$\sin(7x) \tag{3.3}$$

Taylor series:

Force := *taylor*$(F(x), x = 0, 5)$

$$Force := F(0) + \mathrm{D}(F)(0)\,x + \frac{1}{2}\mathrm{D}^{(2)}(F)(0)\,x^2 + \frac{1}{6}\mathrm{D}^{(3)}(F)(0)\,x^3 + \frac{1}{24}\mathrm{D}^{(4)}(F)(0)\,x^4 + \mathrm{O}(x^5) \tag{4.1}$$

Force := *convert*(*Force*, *polynom*)

$$Force := F(0) + \mathrm{D}(F)(0)\,x + \frac{\mathrm{D}^{(2)}(F)(0)\,x^2}{2} + \frac{\mathrm{D}^{(3)}(F)(0)\,x^3}{6} + \frac{\mathrm{D}^{(4)}(F)(0)\,x^4}{24} \tag{4.2}$$

Checking the statement:

sin(theta[1] + theta[2]) = *expand*(sin(theta[1] + theta[2]))

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \tag{5.1}$$

cos(2 theta) = *expand*(cos(2 theta))

$$\cos(2\theta) = 2\cos(\theta)^2 - 1 \tag{5.2}$$

Factor, expand and simplify:

poly := $x^5 - 3x^4 - 23x^3 + 51x^2 + 94x - 120$;

$$poly := x^5 - 3x^4 - 23x^3 + 51x^2 + 94x - 120 \tag{6.1}$$

poly := *factor*(*poly*);

$$poly := (x-1)(x-3)(x-5)(x+4)(x+2) \tag{6.2}$$

simplify$\left(x^2 + 2x + 1 + \frac{1}{x^2 + 2x + 1}\right)$

$$x^2 + 2x + 1 + \frac{1}{(x+1)^2} \tag{6.3}$$

expand$\left(x^2 + 2x + 1 + \frac{1}{x^2 + 2x + 1}\right)$

$$x^2 + 2x + 1 + \frac{1}{x^2 + 2x + 1} \quad \textbf{(6.4)}$$

$expand\left(simplify\left(x^2 + 2x + 1 + \frac{1}{x^2 + 2x + 1}\right)\right)$

$$x^2 + 2x + 1 + \frac{1}{(x+1)^2} \quad \textbf{(6.5)}$$

Solving system of Equation:

For a certain direct current (dc) circuit, Kirchhoff's loop rules give the following four simultanious equations with unknown currents x_1, x_2, x_3 and x_4. Solve the set of currents uto 3-digits.

$$x_1, x_2, x_3$$

$$x_4 \quad \textbf{(7.1)}$$

$Digits := 3$

$$Digits := 3 \quad \textbf{(7.2)}$$

$eq1 := 2\,x[1] + 3\,x[2] - 6\,x[3] - 5\,x[4] = 2;$

$$eq1 := 2\,x_1 + 3\,x_2 - 6\,x_3 - 5\,x_4 = 2 \quad \textbf{(7.3)}$$

$eq2 := x[1] - 2\,x[2] - 4\,x[3] + 2\,x[4] = 0;$

$$eq2 := x_1 - 2\,x_2 - 4\,x_3 + 2\,x_4 = 0 \quad \textbf{(7.4)}$$

$eq3 := 3\,x[1] + 2.5\,x[2] + x[4] = 5;$

$$eq3 := 3\,x_1 + 2.5\,x_2 + x_4 = 5 \quad \textbf{(7.5)}$$

$eq4 := x[2] - 23\,x[3] + 9.3\,x[4] = 7.2;$

$$eq4 := x_2 - 23\,x_3 + 9.3\,x_4 = 7.2 \quad \textbf{(7.6)}$$

$solution := solve(\{eq1, eq2, eq3, eq4\}, \{x[1], x[2], x[3], x[4]\});$

$$solution := \{x_1 = 0.675, x_2 = 0.979, x_3 = -0.0575, x_4 = 0.527\} \quad \textbf{(7.7)}$$

$assign(solution) : x[3];$

$$-0.0575 \quad \textbf{(7.8)}$$

$x[3];$

$$-0.0575 \quad \textbf{(7.9)}$$

Creating plot with data and with formula:

$restart : with(plots) : with(stats[statplots]) :$
$unprotect(time) :$
$time := [0.347, 0.470, 0.519, 0.582, 0.650, 0.674, 0.717, 0.766, 0.823, 0.870, 1.031, 1.193, 1.354,$

1.501, 1.726, 1.873];

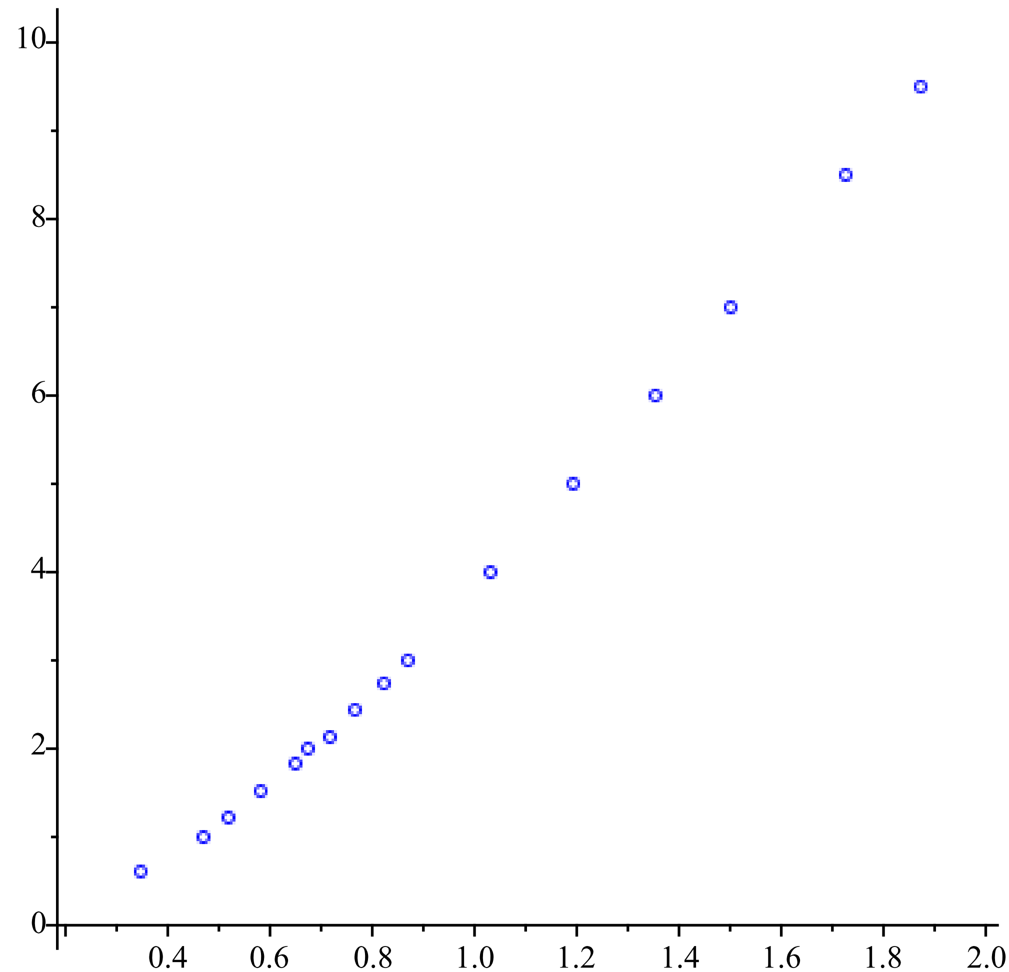
$time := [0.347, 0.470, 0.519, 0.582, 0.650, 0.674, 0.717, 0.766, 0.823, 0.870, 1.031, 1.193, 1.354, 1.501, 1.726, 1.873]$ **(8.1)**

$distance := [0.61, 1.00, 1.22, 1.52, 1.83, 2.00, 2.13, 2.44, 2.74, 3.00, 4.00, 5.00, 6.00, 7.00, 8.50, 9.50];$

$distance := [0.61, 1.00, 1.22, 1.52, 1.83, 2.00, 2.13, 2.44, 2.74, 3.00, 4.00, 5.00, 6.00, 7.00, 8.50, 9.50]$ **(8.2)**

scatterplot command is used to plot the distance vs time data:

$pts := scatterplot(time, distance, symbol = circle, color = blue)$



Theoritical formula:

$y := \left(\frac{v[T]^2}{g}\right) \ln\left(\cosh\left(\frac{g\, t}{v[T]}\right)\right);$

$$y := \frac{v_T^2 \ln\left(\cosh\left(\frac{g\, t}{v_T}\right)\right)}{g} \quad \textbf{(8.3)}$$

$g := 9.81; v[T] := 6.80;$

$$g := 9.81$$

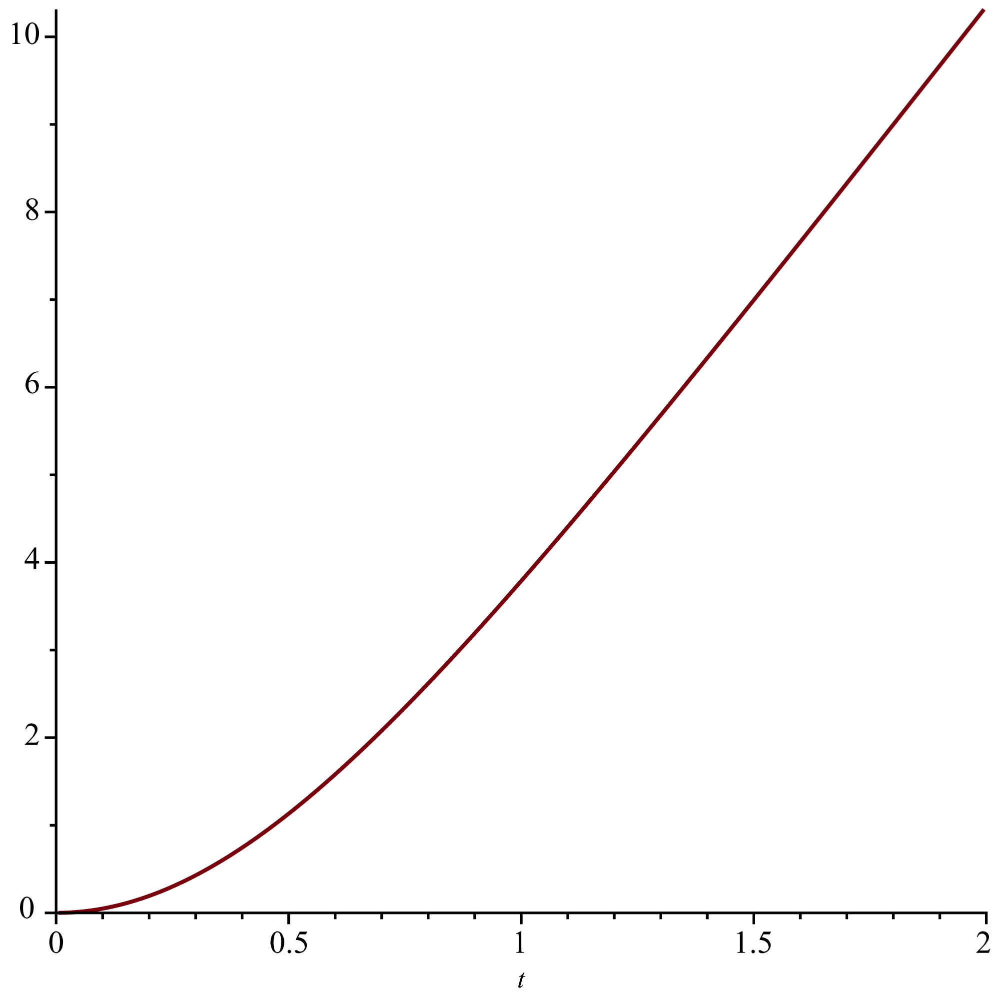
$$v_T := 6.80 \tag{8.4}$$

Maple automatically substitute the values into the formula:

$y := y;$

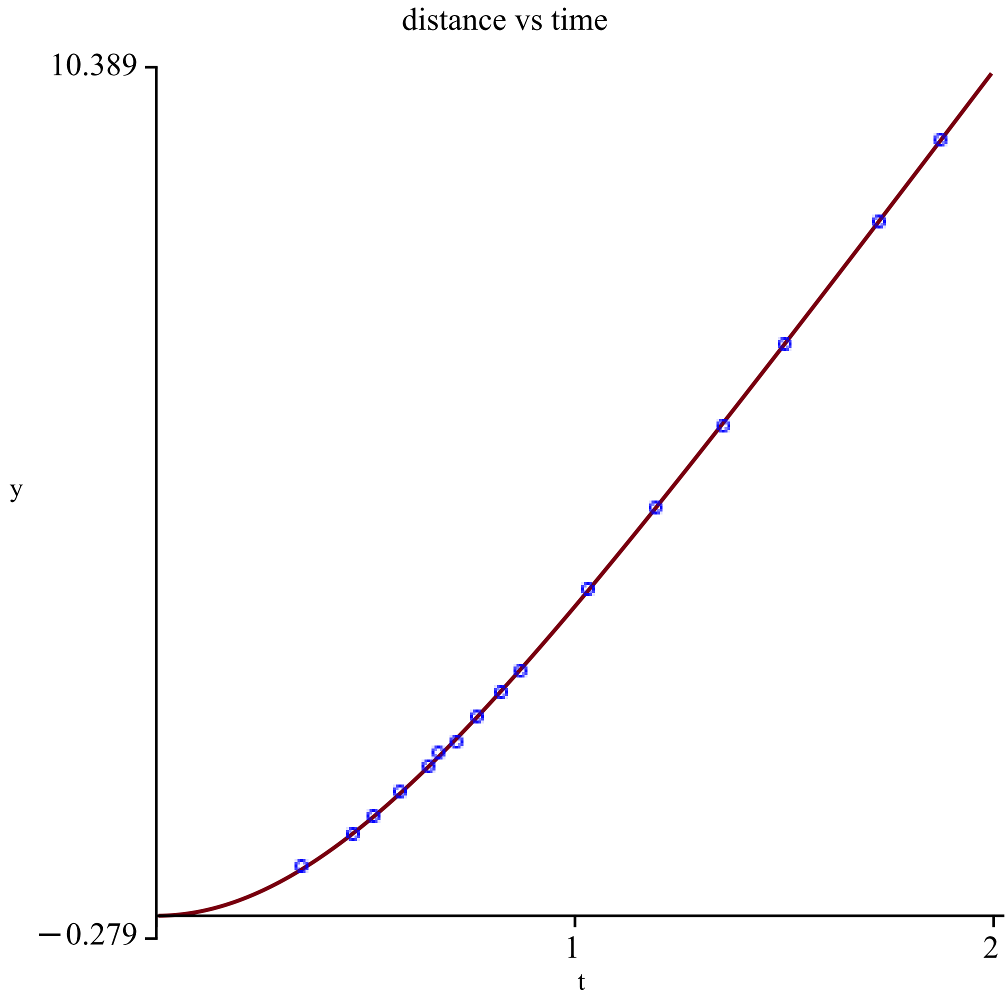
$$y := 4.713557594 \ln(\cosh(1.442647059\, t)) \tag{8.5}$$

$Gr := plot(y, t = 0 ..2)$



To superimpose the two graphs display command is used:

$display(\{pts, Gr\}, tickmarks = [3, 2], labels = [\text{"t"}, \text{"y"}], title = \text{"distance vs time"});$



Solving DE and collecting/combining coefficient and substituting value:

restart;

eqn := *diff*(*u*(*x*, *y*), *x*, *y*) + 2 sin(theta) *diff*(*u*(*x*, *y*), *x*, *x*) + 2 cos(theta) *diff*(*u*(*x*, *y*), *y*, *y*) + 3 sin(theta) *diff*(*u*(*x*, *y*, *t*), *y*, *y*) + 7 cos(theta) *diff*(*u*(*x*, *y*), *x*) = 0;

$$eqn := \frac{\partial^2}{\partial x \partial y} u(x, y) + 2 \sin(\theta) \left(\frac{\partial^2}{\partial x^2} u(x, y)\right) + 2 \cos(\theta) \left(\frac{\partial^2}{\partial y^2} u(x, y)\right) + 3 \sin(\theta) \left(\frac{\partial^2}{\partial y^2} u(x, y, t)\right) + 7 \cos(\theta) \left(\frac{\partial}{\partial x} u(x, y)\right) = 0 \quad \textbf{(9.1)}$$

collect(*eqn*, {sin(theta), cos(theta)});

$$\left(2 \frac{\partial^2}{\partial y^2} u(x, y) + 7 \frac{\partial}{\partial x} u(x, y)\right) \cos(\theta) + \left(2 \frac{\partial^2}{\partial x^2} u(x, y) + 3 \frac{\partial^2}{\partial y^2} u(x, y, t)\right) \sin(\theta) + \frac{\partial^2}{\partial x \partial y} u(x, y) = 0 \quad \textbf{(9.2)}$$

$deqn := y''(x) - y'(x) + 2 = 0;$ $\xrightarrow{\text{solve DE}}$ $y(x) = e^{x} c_1 + 2x + c_2$

above DE is solved by using right-side pane with "solve DE" option and below is solving with command

$dsol := dsolve(deqn, y(x))$

$$dsol := y(x) = e^{x} c_1 + 2x + c_2 \tag{9.3}$$

$y(x)$

$$y(x) \tag{9.4}$$

Assignment of solution to y(x):

$assign(dsol) : y(x);$

$$e^{x} c_1 + 2x + c_2 \tag{9.5}$$

$y(x)$

$$e^{x} c_1 + 2x + c_2 \tag{9.6}$$

automatic substitution if we assign solution before:

$new\_eqn := y(x)^2 + y(x) + 3 = 0;$

$$new\_eqn := \left(e^{x} c_1 + 2x + c_2\right)^2 + e^{x} c_1 + 2x + c_2 + 3 = 0 \tag{9.7}$$

or we can substitute by subs command, let say in $\#y^3(x)$.

$subs\left(y(x) = y(x), y(x)^3\right)$

$$\left(e^{x} c_1 + 2x + c_2\right)^3 \tag{9.8}$$

Another example of subs:

$eqn3 := diff(u(x,y), x) + diff(u(x,y), y);$

$$eqn3 := \frac{\partial}{\partial x} u(x,y) + \frac{\partial}{\partial y} u(x,y) \tag{9.9}$$

$subs(diff(u(x,y), x) = 0, eqn3)$

$$\frac{\partial}{\partial y} u(x,y) \tag{9.10}$$

$eqn4 := diff(u(x,y), x)\ diff(u(x,y), y) + diff(u(x,y), x, x) + diff(u(x,y), y, y);$

$$eqn4 := \left(\frac{\partial}{\partial x} u(x,y)\right)\left(\frac{\partial}{\partial y} u(x,y)\right) + \frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial y^2} u(x,y) \tag{9.11}$$

$subs(diff(u(x,y), y) \cdot diff(u(x,y), x) = 0, eqn4);$

$$\frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial y^2} u(x,y) \tag{9.12}$$

again with changing order of occuring:

$subs(diff(u(x,y), x)\ diff(u(x,y), y) = 0, eqn4);$

$$\frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) \quad \textbf{(9.13)}$$

Using dsubs( ):

restart : *with*(*PDEtools*) :

eqn5 := *diff* (*u*(*x*, *y*), *x*) *diff* (*u*(*x*, *y*), *y*) + *diff* (*u*(*x*, *y*), *x*, *x*) + *diff* (*u*(*x*, *y*), *y*, *y*);

$$eqn5 := \left(\frac{\partial}{\partial x} u(x, y)\right)\left(\frac{\partial}{\partial y} u(x, y)\right) + \frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) \quad \textbf{(9.14)}$$

dsubs(*diff* (*u*(*x*, *y*), *y*) ·*diff* (*u*(*x*, *y*), *x*) = 0, *eqn5*)

$$\frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) \quad \textbf{(9.15)}$$

again with changing order of occuring:

dsubs(*diff* (*u*(*x*, *y*), *x*) *diff* (*u*(*x*, *y*), *y*) = 0, *eqn5*);

$$\frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) \quad \textbf{(9.16)}$$

Using subs( ) in MTM:

restart : *with*(*MTM*) :

eqn6 := *diff* (*u*(*x*, *y*), *x*) *diff* (*u*(*x*, *y*), *y*) + *diff* (*u*(*x*, *y*), *x*, *x*) + *diff* (*u*(*x*, *y*), *y*, *y*);

$$eqn6 := \left(\frac{\partial}{\partial x} u(x, y)\right)\left(\frac{\partial}{\partial y} u(x, y)\right) + \frac{\partial^x}{\partial x^x} u(x, y) + \frac{\partial^y}{\partial y^y} u(x, y) \quad \textbf{(9.17)}$$

subs(*eqn6*, *diff* (*u*(*x*, *y*), *y*) ·*diff* (*u*(*x*, *y*), *x*), 0);

$$\frac{\partial^x}{\partial x^x} u(x, y) + \frac{\partial^y}{\partial y^y} u(x, y) \quad \textbf{(9.18)}$$

subs(*eqn6*, *diff* (*u*(*x*, *y*), *x*) ·*diff* (*u*(*x*, *y*), *y*), 0);

$$\frac{\partial^x}{\partial x^x} u(x, y) + \frac{\partial^y}{\partial y^y} u(x, y) \quad \textbf{(9.19)}$$

Numerical Solution of ODE

restart : *with*(*plots*) :

eq := *diff* (*y*(*t*), *t*, *t*) + *y*(*t*) + 0.25 *y*(*t*)^2 = 0.55 sin(0.1 *t*);

$$eq := \frac{d^2}{dt^2} y(t) + y(t) + 0.25\, y(t)^2 = 0.55 \sin(0.1\, t) \quad \textbf{(10.1)}$$

ic := *y*(0) = 1, D(*y*)(0) = 0;

$$ic := y(0) = 1, D(y)(0) = 0 \quad \textbf{(10.2)}$$

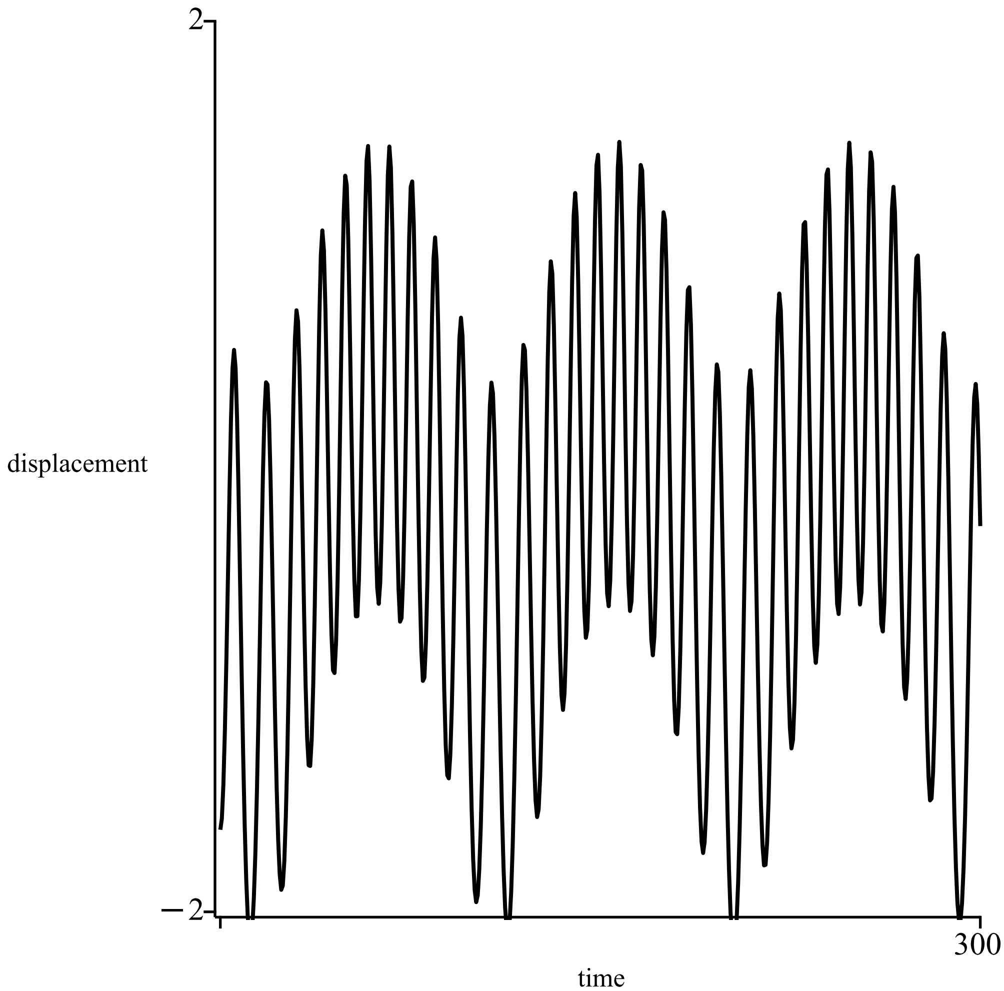
sol := *dsolve*({*eq*, *ic*}, *y*(*t*), *type* = *numeric*, *output* = *listprocedure*);

$sol := \left[t = \textbf{proc}(t)\ \ldots\ \textbf{end proc}, y(t) = \textbf{proc}(t)\ \ldots\ \textbf{end proc}, \frac{d}{dt}\, y(t) = \textbf{proc}(t) \right.$ **(10.3)**

...

end proc]

odeplot(*sol*, [*t*, *y*(*t*)], 100 ..300, *numpoints* = 500, *axes* = *framed*, *tickmarks* = [2, 2], *labels* = ["time", "displacement"], *view* = [100 ..300, −2 ..2], *color* = *black*);



Working with system

restart : *with*(*DEtools*) :

$a[r] := 2 : b[r] := 0.01 : a[f] := 1 : b[f] := 0.01 :$

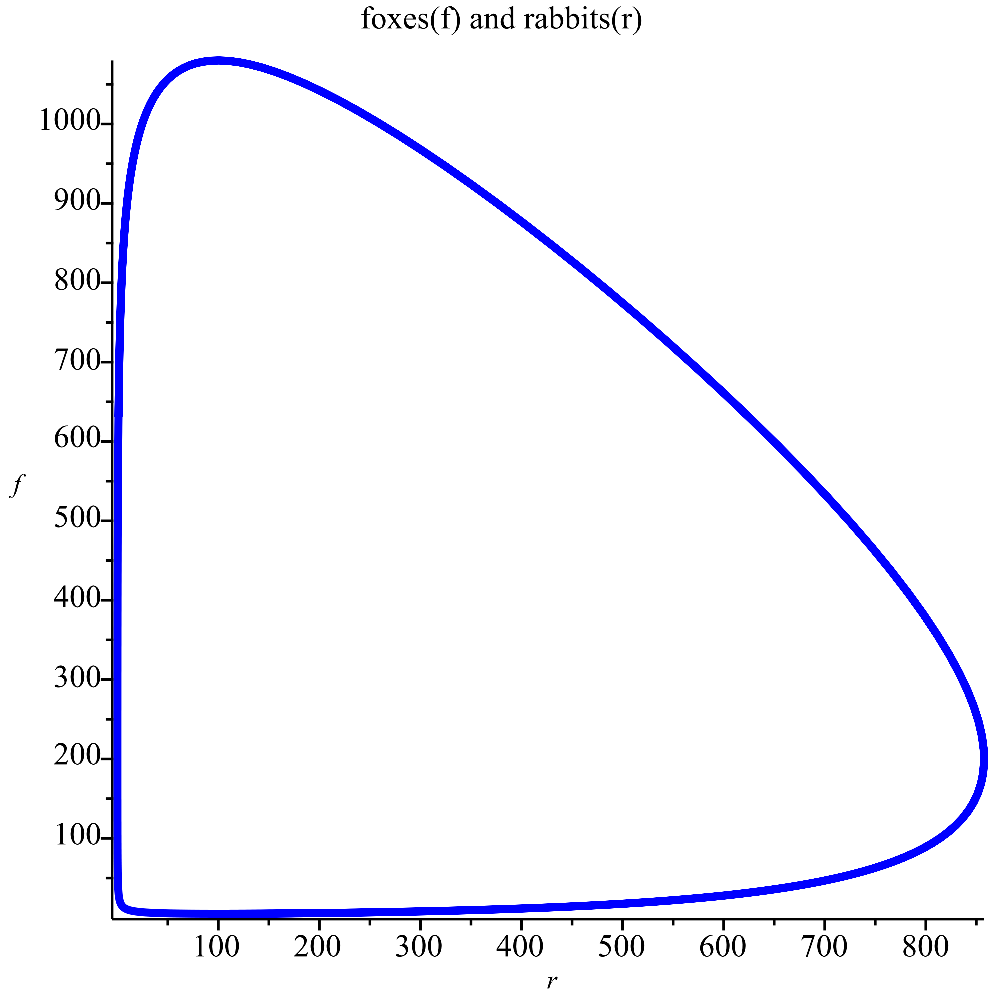
Rabbits-foxes system

RF := *diff*(*r*(*t*), *t*) = *a*[*r*]·*r*(*t*) − *b*[*r*]·*r*(*t*)·*f*(*t*), *diff*(*f*(*t*), *t*) = −*a*[*f*]·*f*(*t*) + *b*[*f*]·*r*(*t*)·*f*(*t*);

$$RF := \frac{d}{dt}\, r(t) = 2\, r(t) - 0.01\, r(t)\, f(t), \frac{d}{dt}\, f(t) = -f(t) + 0.01\, r(t)\, f(t) \quad \textbf{(11.1)}$$

need to put system in a list using [ ]

phaseportrait([*RF*], [*r*(*t*), *f*(*t*)], *t* = 0 ..10, [[*r*(0) = 100, *f*(0) = 5]], *stepsize* = 0.01, *linecolor* = *blue*, *arrows* = *NONE*, *title* = "foxes(f) and rabbits(r)");



Summition and Animation

restart : *with*(*plots*) :

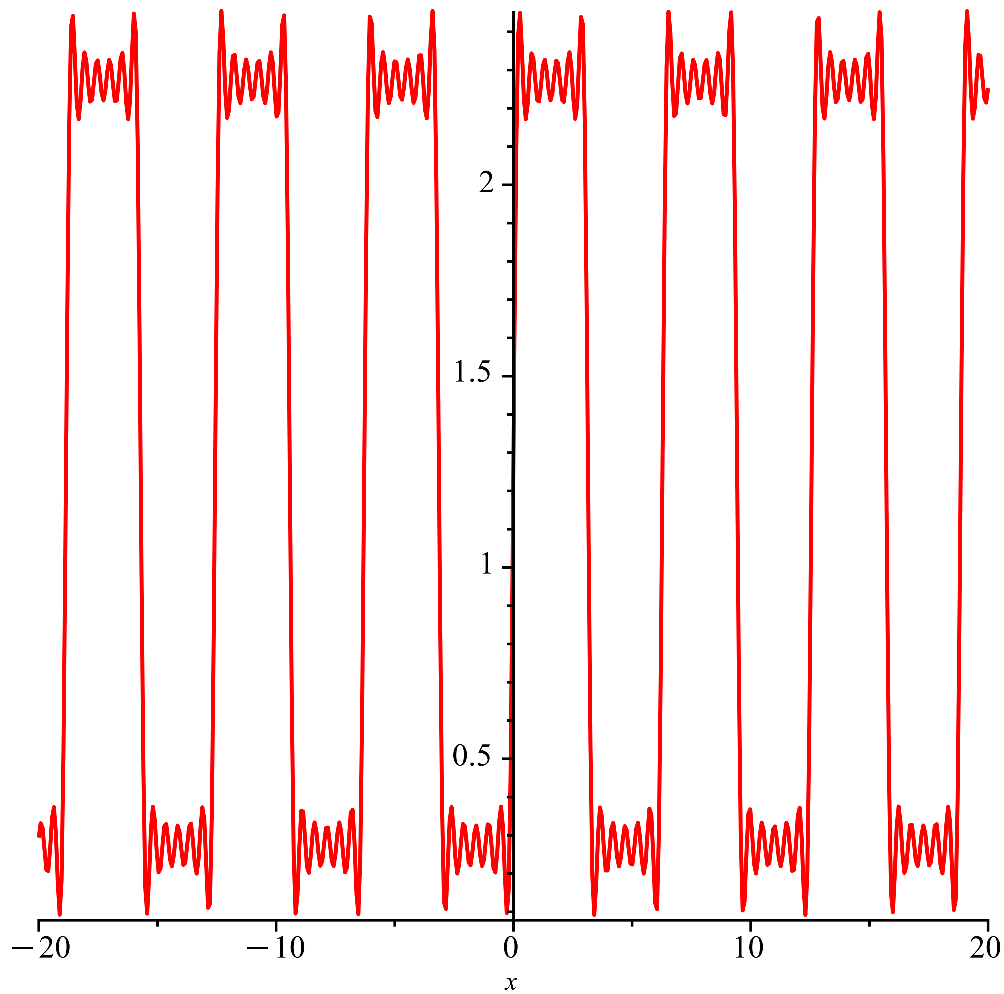
$c := 1 : N := 5 :$

unassign('*x*','*t*') :

$$\text{psi} := \frac{4}{\text{Pi}}\left(1 + sum\left(\frac{\sin((2\,n+1)\,(x - c\,t))}{(2\,n+1)}, n = 0..N\right)\right)$$

$$\psi := \frac{1}{\pi}\left(4\left(1 - \sin(t - x) - \frac{\sin(3\,t - 3\,x)}{3} - \frac{\sin(5\,t - 5\,x)}{5} - \frac{\sin(7\,t - 7\,x)}{7} - \frac{\sin(9\,t - 9\,x)}{9} - \frac{\sin(11\,t - 11\,x)}{11}\right)\right) \quad \textbf{(12.1)}$$

animate(psi, *x* = −20 ..20, *t* = 0 ..10, *frames* = 50, *numpoints* = 500);



Change of variables:

$with(PDEtools)$:
$PDE := diff(f(x,y),x) + g(x,y) + diff(f(x,y),y) = 0;$

$$PDE := \frac{\partial}{\partial x} f(x,y) + g(x,y) + \frac{\partial}{\partial y} f(x,y) = 0 \quad \textbf{(13.1)}$$

$tr := \{x = r + s, y = r - s\};$

$$tr := \{x = r + s, y = r - s\} \quad \textbf{(13.2)}$$

$dchange(tr, PDE);$

$$\frac{\partial}{\partial r} f(r,s) + g(r,s) = 0 \quad \textbf{(13.3)}$$

Tell Maple you are looking at g as a "known" function:

$dchange(tr, PDE, known = g);$

$$\frac{\partial}{\partial r} f(r,s) + g(r+s, r-s) = 0 \quad \textbf{(13.4)}$$

The next example demonstrates the reduction of the number of independent variables inside a differential operator:

$$> L := f \to x \left(\frac{\partial}{\partial y} f\right) - y \left(\frac{\partial}{\partial x} f\right)$$

$$L := f \to x \left(\frac{\partial}{\partial y} f\right) - y \left(\frac{\partial}{\partial x} f\right) \quad \textbf{(13.5)}$$

$$> tr := \left\{x = s\, r^{\frac{1}{2}}, y = s\,(1-r)^{\frac{1}{2}}\right\}$$

$$tr := \left\{x = s\sqrt{r}, y = s\sqrt{1-r}\right\} \quad \textbf{(13.6)}$$

The transformation tr reduces the number of differentiation variables:

$$> `L'` = dchange(tr, L, simplify)$$

$$L' = \left(f \to -2\sqrt{r}\sqrt{1-r}\left(\frac{\partial}{\partial r} f\right)\right) \quad \textbf{(13.7)}$$

This example demonstrates the use of the differential operator d/dy alone:

$$> d[y] := f \to \frac{\partial}{\partial y} f$$

$$d_y := f \to \frac{\partial}{\partial y} f \quad \textbf{(13.8)}$$

$$> dchange(tr, d[y], normal)$$

$$f \to -\frac{\sqrt{1-r}\left(2\left(\frac{\partial}{\partial r} f\right) r - \left(\frac{\partial}{\partial s} f\right) s\right)}{s} \quad \textbf{(13.9)}$$

if..then..else.. end if

```
> a := 10; b := 15;
```

$$a := 10$$

$$b := 15 \quad \textbf{(14.1)}$$

Since a (3) is not greater than b (5), the b branch is followed.

```
> if (a > b) then a else b end if;
```

$$15 \quad \textbf{(14.2)}$$

FAIL is used by the Boolean logic to mean an unknown or undetermined value. A FAIL has the same effect as false.

```
> if 'FAIL' then 3 else 5 end if;
```

$$5 \quad \textbf{(14.3)}$$

```
> c := 12;
```

$$c := 12 \quad \textbf{(14.4)}$$

In the following equation there are two nested selections. Since a<b is true and a<c is false, d is assigned the calculated value a*c.

```
> if a < b then
    if a < c then d := a*b else d := a*c end if
  end if;
  d
```

$$150 \qquad (14.5)$$

```
> if a < b then
    if a < c then d := a*b else d := a*c end if
  fi;
  d
```

$$150 \qquad (14.6)$$

you can use "fi" in place of "end if" as in above.

use of elif

$a := 3;$

$$a := 3 \qquad (15.1)$$

```
> if a = 1 then print(first)
  elif a = 2 then print(second)
  elif a = 3 then print(third)
  end if;
```

$$third \qquad (15.2)$$

```
> s := String(a, if a = 1 then "st"
                 elif a = 2 then "nd"
                 elif a = 3 then "rd"
                 else "th" end if);
```

$$s := \text{"3rd"} \qquad (15.3)$$

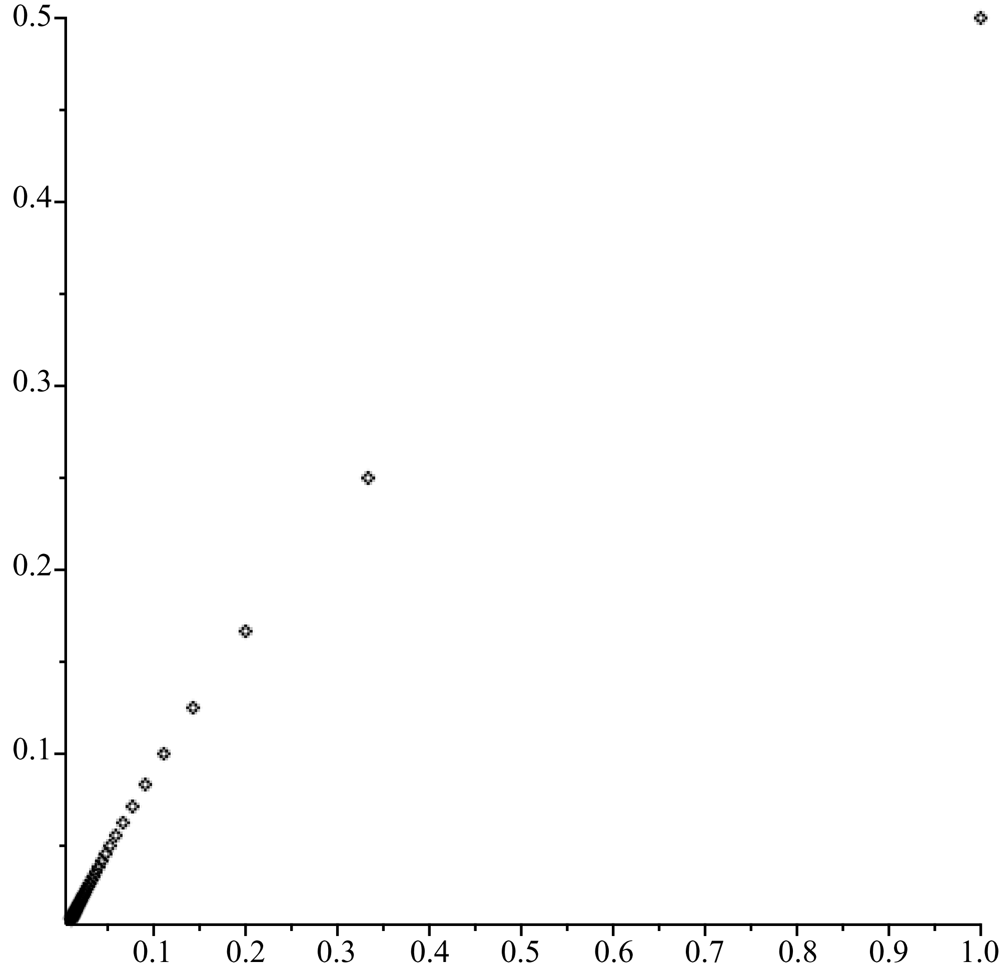
```
>
```

Point plot and sequence

$restart : with(plots) :$

$SEQ := seq\left(\frac{1}{n}, n = 1 ..100\right) :$

$pointplot([SEQ])$



Derivation and Solution of Sine-Gordon Equation

restart : *with*(*plots*) :

eq1 := *m*·*l*·*diff* (psi(*x*, *t*), *t*, *t*) + *m*·*g*·sin(psi(*x*, *t*)) −*k*·(psi(*x* + *h*, *t*) − psi(*x*, *t*)) + *k*·(psi(*x*, *t*) − psi(*x* − *h*, *t*)) = 0;

$$eq1 := m\, l \left(\frac{\partial^2}{\partial t^2}\, \psi(x,t)\right) + m\, g \sin\left(\psi(x,t)\right) - k \left(\psi(x+h,t) - \psi(x,t)\right) + k \left(\psi(x,t) - \psi(x-h,t)\right) = 0 \qquad \textbf{(17.1)}$$

eq2 := *expand*(*eq1* / (*m* * *g*));

$$eq2 := \frac{l \left(\frac{\partial^2}{\partial t^2}\, \psi(x,t)\right)}{g} + \sin\left(\psi(x,t)\right) - \frac{k\, \psi(x+h,t)}{m\, g} + \frac{2\, k\, \psi(x,t)}{m\, g} - \frac{k\, \psi(x-h,t)}{m\, g} = 0 \qquad \textbf{(17.2)}$$

eq3 := *taylor*(*lhs*(*eq2*), *h* = 0, 3) = *rhs*(*eq2*);

$$eq3 := \frac{l\left(\frac{\partial^2}{\partial t^2}\psi(x,t)\right)}{g} + \sin(\psi(x,t)) - \frac{k\,\mathrm{D}_{1,1}(\psi)(x,t)}{m\,g}h^2 + \mathrm{O}(h^3) = 0 \quad \textbf{(17.3)}$$

$eq4 := convert(eq3, polynom);$

$$eq4 := \frac{l\left(\frac{\partial^2}{\partial t^2}\psi(x,t)\right)}{g} + \sin(\psi(x,t)) - \frac{k\,\mathrm{D}_{1,1}(\psi)(x,t)\,h^2}{m\,g} = 0 \quad \textbf{(17.4)}$$

Substitution in group:

$$SGeq := subs\left(\left\{l\cdot diff(\mathrm{psi}(x,t),t,t)/g = diff(\mathrm{psi}(X,T),T,T), \frac{k\cdot h^2\cdot \mathrm{D}[1,1](\mathrm{psi})(x,t)}{(m\cdot g)} = diff(\mathrm{psi}(X,T),X,X), \sin(\mathrm{psi}(x,t)) = \sin(\mathrm{psi}(X,T))\right\}, eq4\right);$$

$$SGeq := \frac{\partial^2}{\partial T^2}\psi(X,T) + \sin(\psi(X,T)) - \frac{\partial^2}{\partial X^2}\psi(X,T) = 0 \quad \textbf{(17.5)}$$

$$\mathrm{psi}(X,T) := 4\cdot\arctan\left(\exp\left(\frac{(X - c\cdot T)}{\mathrm{sqrt}(1-c^2)}\right)\right);$$

$$\psi := (X,T) \mapsto 4\cdot\arctan\left(\mathrm{e}^{\frac{X-c\cdot T}{\sqrt{-c^2+1}}}\right) \quad \textbf{(17.6)}$$

$SGeq2 := expand(SGeq);$

$$SGeq2 := \frac{4\,c^2\,\mathrm{e}^{\frac{X}{\sqrt{-c^2+1}}}}{(-c^2+1)\,\mathrm{e}^{\frac{c\,T}{\sqrt{-c^2+1}}}\left(\frac{\left(\mathrm{e}^{\frac{X}{\sqrt{-c^2+1}}}\right)^2}{\left(\mathrm{e}^{\frac{c\,T}{\sqrt{-c^2+1}}}\right)^2}+1\right)} - \frac{8\,c^2\left(\mathrm{e}^{\frac{X}{\sqrt{-c^2+1}}}\right)^3}{(-c^2+1)\left(\mathrm{e}^{\frac{c\,T}{\sqrt{-c^2+1}}}\right)^3\left(\frac{\left(\mathrm{e}^{\frac{X}{\sqrt{-c^2+1}}}\right)^2}{\left(\mathrm{e}^{\frac{c\,T}{\sqrt{-c^2+1}}}\right)^2}+1\right)^2} \quad \textbf{(17.7)}$$

$$+\frac{8\,\mathrm{e}^{\frac{X}{\sqrt{-c^2+1}}}}{\left(\frac{\left(\mathrm{e}^{\frac{X}{\sqrt{-c^2+1}}}\right)^2}{\left(\mathrm{e}^{\frac{cT}{\sqrt{-c^2+1}}}\right)^2}+1\right)^2 \mathrm{e}^{\frac{cT}{\sqrt{-c^2+1}}}}-\frac{4\,\mathrm{e}^{\frac{X}{\sqrt{-c^2+1}}}}{\mathrm{e}^{\frac{cT}{\sqrt{-c^2+1}}}\left(\frac{\left(\mathrm{e}^{\frac{X}{\sqrt{-c^2+1}}}\right)^2}{\left(\mathrm{e}^{\frac{cT}{\sqrt{-c^2+1}}}\right)^2}+1\right)}$$

$$-\frac{4\,\mathrm{e}^{\frac{X}{\sqrt{-c^2+1}}}}{\left(-c^2+1\right)\mathrm{e}^{\frac{cT}{\sqrt{-c^2+1}}}\left(\frac{\left(\mathrm{e}^{\frac{X}{\sqrt{-c^2+1}}}\right)^2}{\left(\mathrm{e}^{\frac{cT}{\sqrt{-c^2+1}}}\right)^2}+1\right)}$$

$$+\frac{8\left(\mathrm{e}^{\frac{X}{\sqrt{-c^2+1}}}\right)^3}{\left(-c^2+1\right)\left(\mathrm{e}^{\frac{cT}{\sqrt{-c^2+1}}}\right)^3\left(\frac{\left(\mathrm{e}^{\frac{X}{\sqrt{-c^2+1}}}\right)^2}{\left(\mathrm{e}^{\frac{cT}{\sqrt{-c^2+1}}}\right)^2}+1\right)^2}=0$$

radnormal(%);

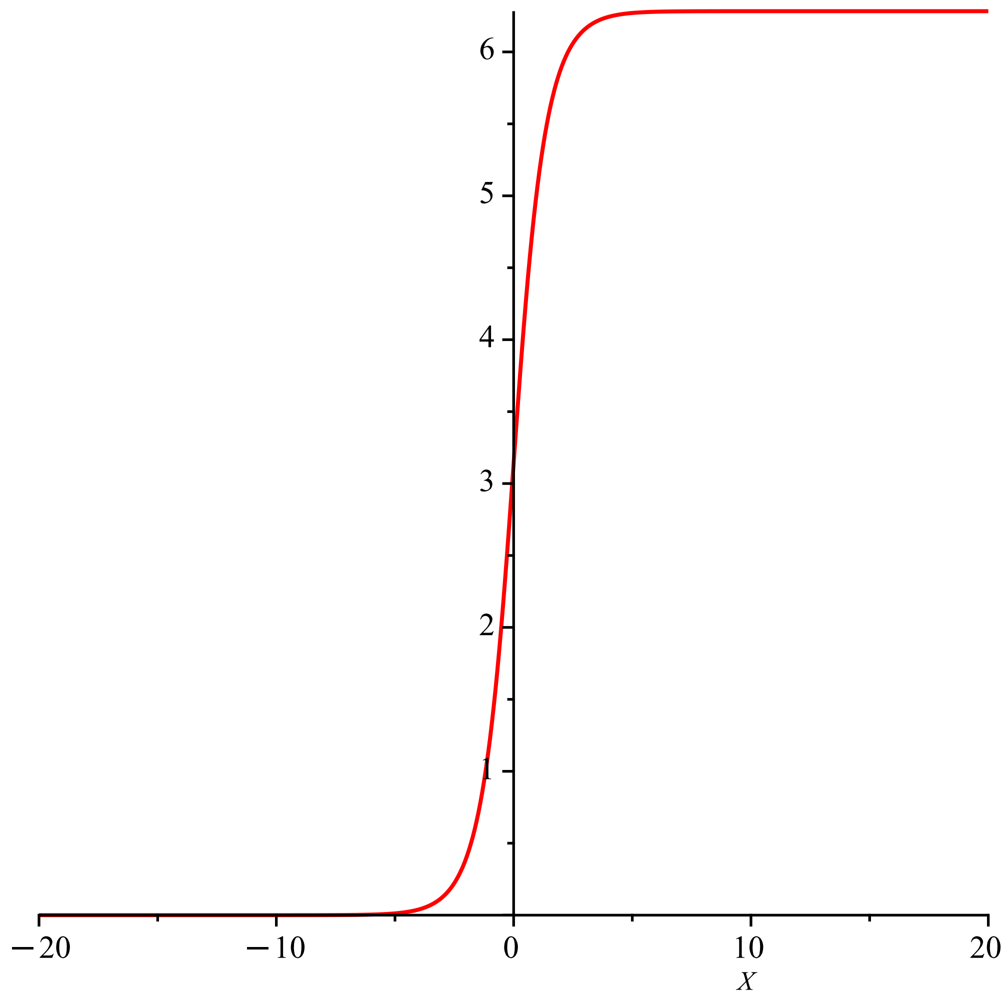
$$0=0 \tag{17.8}$$

Animation at c=0.5

$c := 0.5;$

$$c := 0.5 \tag{17.9}$$

animate(psi(*X*, *T*), *X*=−20 ..20, *T*=0 ..20, *frames*=50, *numpoints*=500);



Piecewise function

$r := piecewise(x < 2, -2000, -2015);$

$$r := \begin{cases} -2000 & x < 2 \\ -2015 & otherwise \end{cases} \tag{18.1}$$

$s := piecewise(\mathrm{abs}(V[1](t)) < 1.5, -2200, 2000)$

$$s := \begin{cases} -2200 & |V_1(t)| < 1.5 \\ 2000 & \mathit{otherwise} \end{cases} \tag{18.2}$$

Breaking piecewise function:

$assume(a < b, b < c)$

$eq2 := piecewise(a < x \textbf{ and } x < b, 1, b < x \textbf{ and } x < c, 2);$

$$eq2 := \begin{cases} 1 & a\sim < x < b\sim \\ 2 & b\sim < x < 0.5 \end{cases} \tag{18.3}$$

$convert(eq2, piecewise, x);$

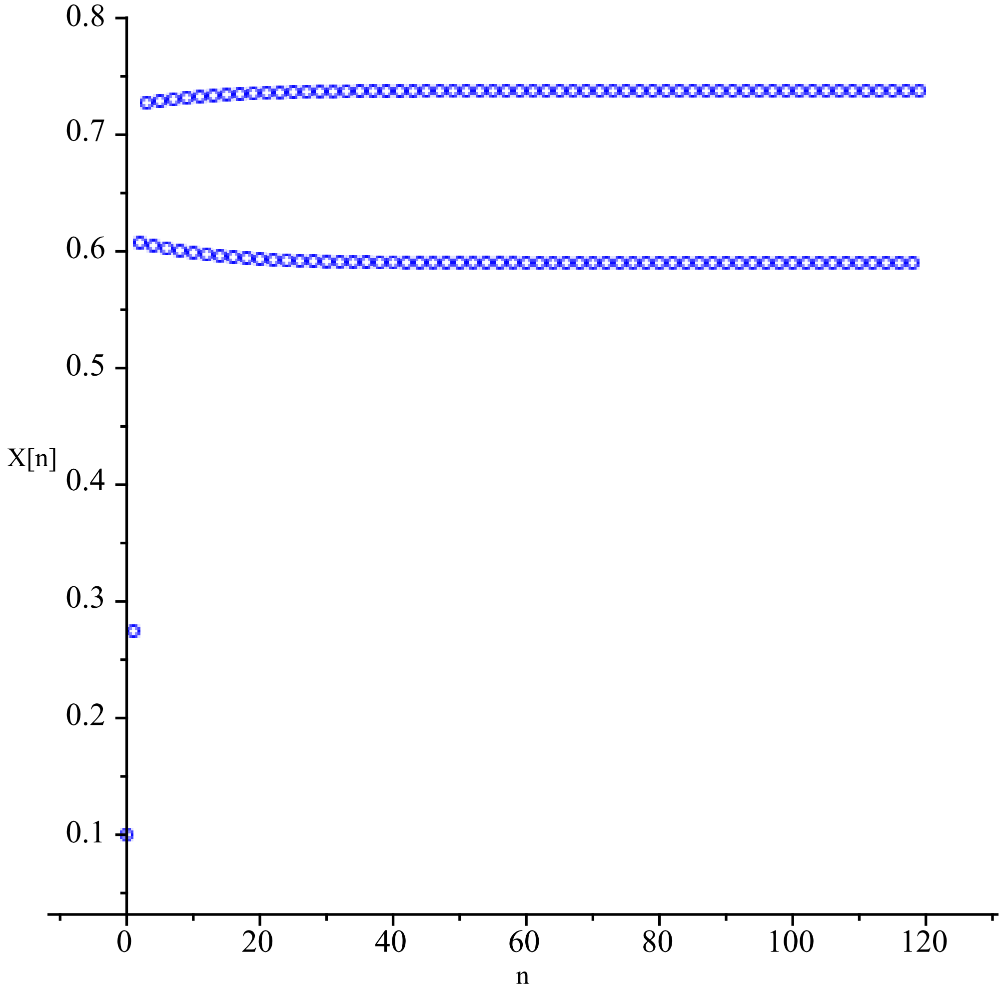
$$\begin{cases} 0 & x \le a\sim \\ 1 & x < b\sim \\ 0 & x = b\sim \\ 2 & x < 0.5 \\ 0 & 0.5 \le x \end{cases} \tag{18.4}$$

The logistic map

```
restart : with(stats[statplots]) :
a := 3.05 : N := 119 : X[0] := 0.1 :
```

Defining sequence with 119 terms:

```
for n from 0 to N do
 X[n + 1] := a·X[n]·(1 − X[n]);
 od:
Xpoints := [seq(X[n], n = 0 .. N) ] :
npoints := [seq(m, m = 0 .. N) ] :
scatterplot(npoints, Xpoints, symbol = circle, color = blue, labels = ["n", "X[n]"]);
```



Predator-prey Map

restart : *with*(*stats*[*statplots*]) :

$a := 3 : b := 3.5 : N := 2000 : X[0] := 0.1 : Y[0] := 0.2 :$

Defining sequence with 119 terms:

for n from 0 to N do

$X[n+1] := a \cdot X[n] \cdot (1 - X[n] - Y[n]);$

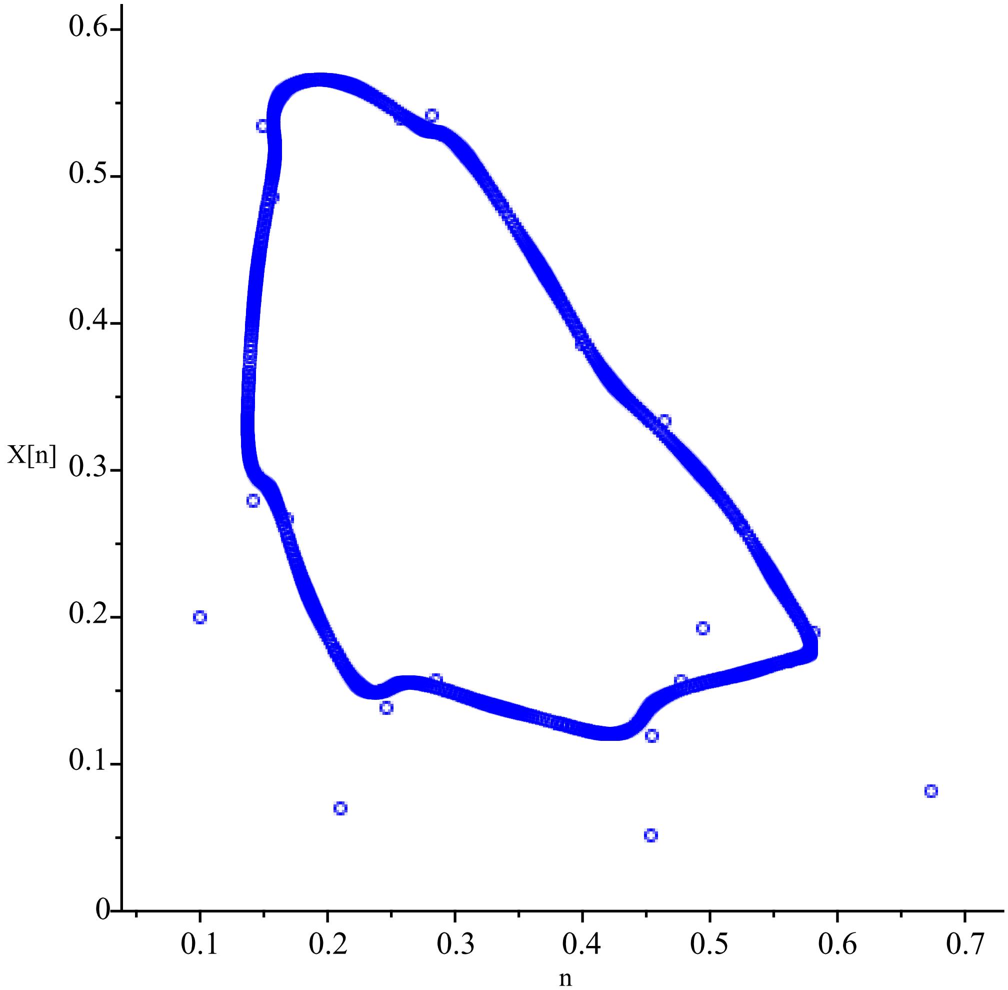
$Y[n+1] := b \cdot X[n] \cdot Y[n]$

od:

Xpoints := [*seq*($X[n]$, $n = 0 .. N$)] :

Ypoints := [*seq*($Y[n]$, $n = 0 .. N$)] :

scatterplot(*Xpoints*, *Ypoints*, *symbol* = *circle*, *color* = *blue*, *labels* = ["n", "X[n]"]);



Damped Harmonic Oscillator Solution

restart : *unprotect*([theta, gamma, omega])

Using theta, gamma and omega as function: (unprotect them before using)

alias(theta = *y*, gamma = *g*, theta[0] = *A*, omega[0] = *w*) :

$osc := diff(y(t), t, t) + 2 \cdot g \cdot diff(y(t), t) + w^2 \cdot y(t) = 0;$

$$osc := \frac{d^2}{dt^2}\,\theta(t) + 2\,\gamma\left(\frac{d}{dt}\,\theta(t)\right) + \omega_0^2\,\theta(t) = 0 \quad \textbf{(21.1)}$$

$soln := dsolve(\{osc, y(0) = A, D(y)(0) = 0\}, y(t), method = laplace);$

$$soln := \theta(t) = e^{-t\gamma}\,\theta_0\left(\cosh\left(t\sqrt{\gamma^2 - \omega_0^2}\right) + \frac{\gamma \sinh\left(t\sqrt{\gamma^2 - \omega_0^2}\right)}{\sqrt{\gamma^2 - \omega_0^2}}\right) \quad \textbf{(21.2)}$$

$collect(soln, [\exp, A, \cos]);$

$$\theta(t) = e^{-t\gamma}\theta_0 \left(\cosh\left(t\sqrt{\gamma^2 - \omega_0^2}\right) + \frac{\gamma \sinh\left(t\sqrt{\gamma^2 - \omega_0^2}\right)}{\sqrt{\gamma^2 - \omega_0^2}} \right) \quad \textbf{(21.3)}$$

Roots-eigenvalues

restart : *with*(*linalg*) :
A := *array*([[*a*, *b*], [*c*, *d*]]);

$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \textbf{(22.1)}$$

Obtaining characteristic polynimial:

cp := *charpoly*(*A*, lambda);

$$cp := a\,d - a\,\lambda - b\,c - \lambda\,d + \lambda^2 \quad \textbf{(22.2)}$$

cp := *collect*(*cp*, lambda);

$$cp := \lambda^2 + (-a - d)\,\lambda + a\,d - b\,c \quad \textbf{(22.3)}$$

cpsubs := *subs*({−*a*−*d*=*p*,−*b* * *c*=*q*−*a* * *d*}, *cp*) =0;

$$cpsubs := \lambda^2 + p\,\lambda + q = 0 \quad \textbf{(22.4)}$$

sol := *solve*(*cpsubs*, lambda);

$$sol := -\frac{p}{2} + \frac{\sqrt{p^2 - 4\,q}}{2}, -\frac{p}{2} - \frac{\sqrt{p^2 - 4\,q}}{2} \quad \textbf{(22.5)}$$

lambda[1] := *sol*[1];
lambda[2] := *sol*[2];

$$\lambda_1 := -\frac{p}{2} + \frac{\sqrt{p^2 - 4\,q}}{2}$$

$$\lambda_2 := -\frac{p}{2} - \frac{\sqrt{p^2 - 4\,q}}{2} \quad \textbf{(22.6)}$$

Tangent field ploting

$restart : with(DEtools) :$

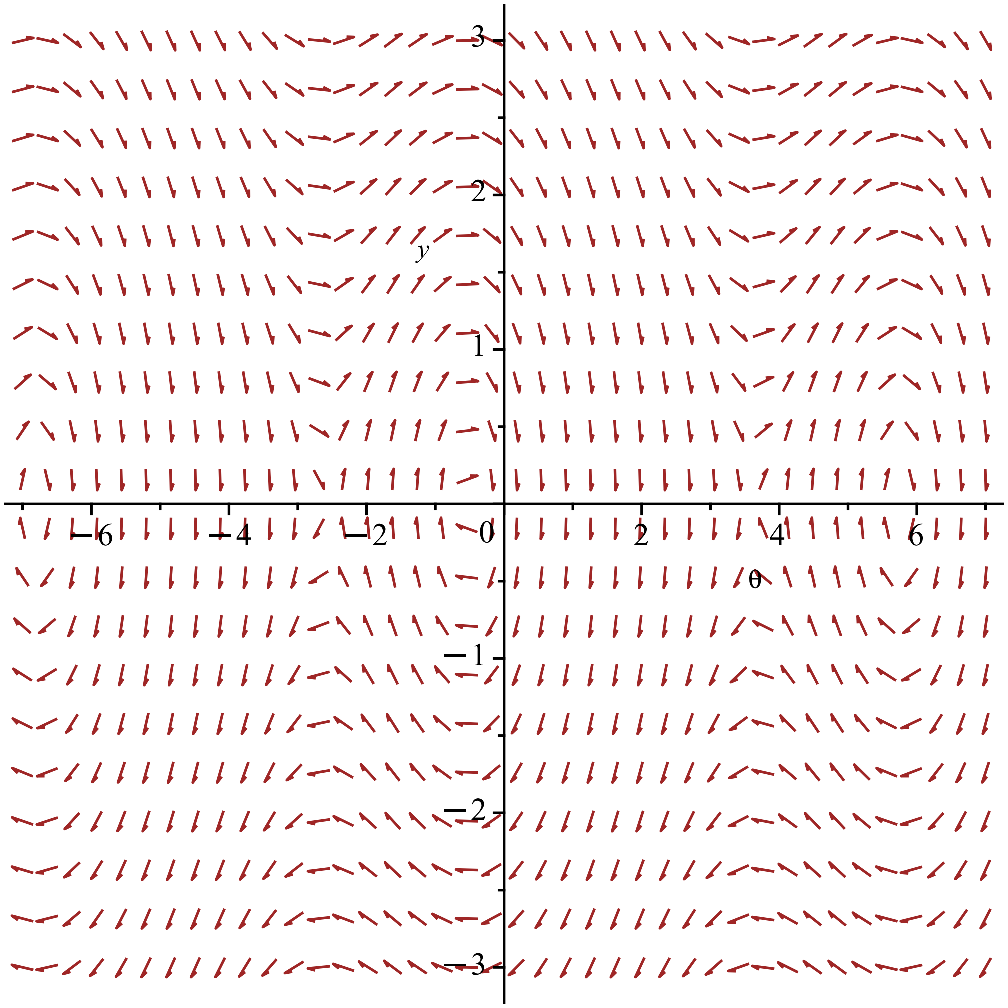
$\text{omega} := 1 :$

$simpend := diff(\text{theta}(t), t) = y(t), diff(y(t), t) = -\text{omega} - 2 \cdot \sin(\text{theta}(t));$

$$simpend := \frac{d}{dt}\,\theta(t) = y(t), \frac{d}{dt}\,y(t) = -1 - 2\sin(\theta(t)) \quad \textbf{(23.1)}$$

The **dfieldplot** command is used to produce the tangent field. The range of t, y and θ are specified as well as the number of grid points to be selected in the horizontal and vertical directions. The default number is [20,20].

$dfieldplot([simpend], [\text{theta}(t), y(t)], t = -4..4, y = -3..3, \text{theta} = -7..7, dirgrid = [40, 20]);$



Competing armies with coefficient collection

restart : *with*(*plots*) : *alias*($C[1]=x$, $C[2]=y$) :
xprime := alpha·x − beta·x·y;

$$xprime := -\beta\, C_1\, C_2 + \alpha\, C_1 \quad \textbf{(24.1)}$$

yprime := (alpha + 1) ·y − gamma·beta·x·y;

$$yprime := (\alpha + 1)\, C_2 - \gamma\beta\, C_1\, C_2 \quad \textbf{(24.2)}$$

eq := *simplify*(*yprime* / *xprime*);

$$eq := \frac{C_2\,(\gamma\beta\, C_1 - \alpha - 1)}{C_1\,(\beta\, C_2 - \alpha)} \quad \textbf{(24.3)}$$

sol := *solve*({*xprime*, *yprime*}, {x, y});

$$sol := \{C_1 = 0, C_2 = 0\}, \left\{C_1 = \frac{\alpha + 1}{\gamma\beta}, C_2 = \frac{\alpha}{\beta}\right\} \quad \textbf{(24.4)}$$

$assign(sol[2]);$
$x := x; y := y;$

$$C_1 := \frac{\alpha + 1}{\gamma \beta}$$
$$C_2 := \frac{\alpha}{\beta} \tag{24.5}$$

$x := x + u; y := y + v;$

$$C_1 := \frac{\alpha + 1}{\gamma \beta} + u$$
$$C_2 := \frac{\alpha}{\beta} + v \tag{24.6}$$

$uprime := simplify(collect(xprime, [u, v]));$

$$uprime := -\frac{v\,(\beta\,\gamma\,u + \alpha + 1)}{\gamma} \tag{24.7}$$

Picking coefficient with coeff **function:**

$a := coeff(uprime, u);$
$a := subs(v = 0, a);$

$$a := -\beta\,v$$
$$a := 0 \tag{24.8}$$

$b := coeff(uprime, v);$
$b := subs(u = 0, b);$

$$b := -\frac{\beta\,\gamma\,u + \alpha + 1}{\gamma}$$
$$b := -\frac{\alpha + 1}{\gamma} \tag{24.9}$$

$vprime := simplify(collect(yprime, [u, v]));$

$$vprime := -\gamma\,(\beta\,v + \alpha)\,u \tag{24.10}$$

$c := coeff(vprime, u);$
$c := subs(v = 0, c);$

$$c := -\gamma\,(\beta\,v + \alpha)$$
$$c := -\gamma\alpha \tag{24.11}$$

$d := coeff(vprime, v);$
$d := subs(u = 0, d);$

$$d := -\beta\,\gamma\,u$$
$$d := 0 \tag{24.12}$$

$p := simplify(-(a+d));$

$$p := 0 \quad \textbf{(24.13)}$$

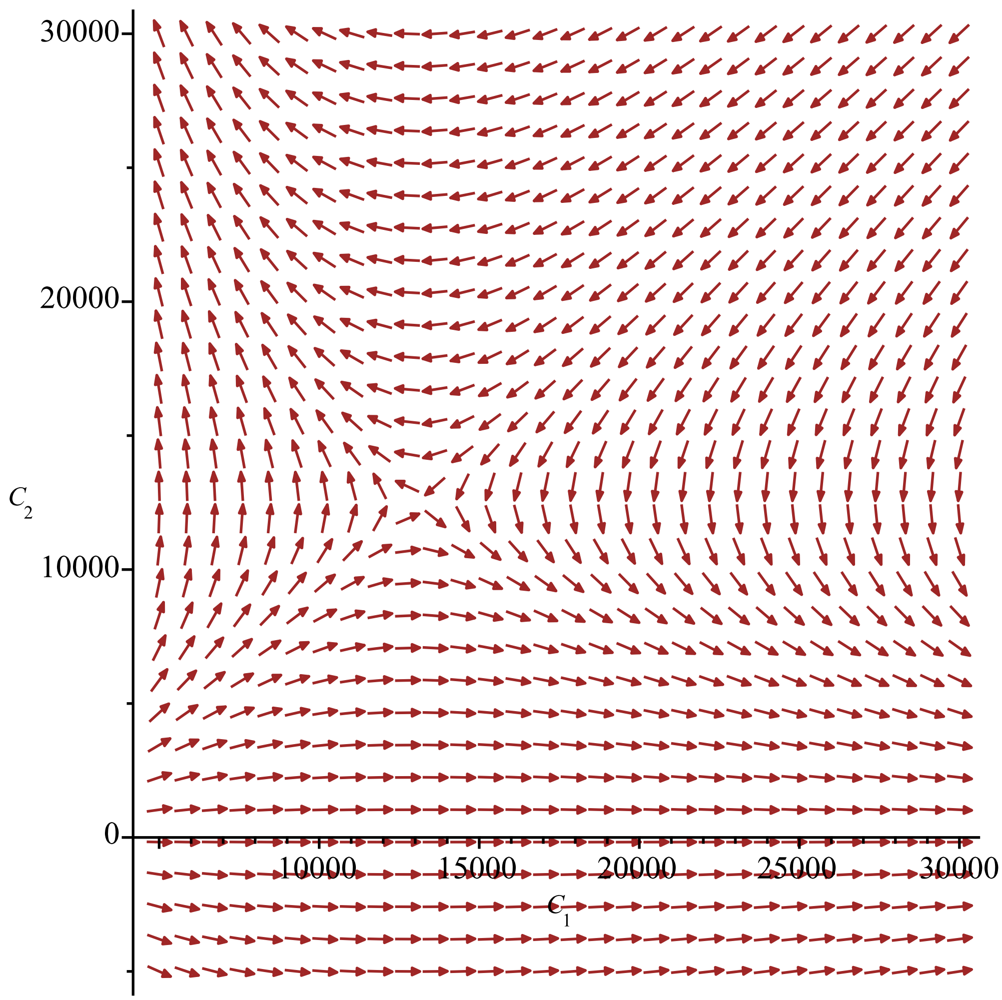
$q := simplify(a \cdot d - b \cdot c);$

$$q := -(\alpha+1)\,\alpha \quad \textbf{(24.14)}$$

$restart : with(DEtools) : unprotect(\text{gamma}) :$
$\text{alpha} := 5 : \text{gamma} := 1.15 : \text{beta} := 1/2500 :$
$alias(C[1]=x, C[2]=y) :$
$armies := diff(x(t),t) = \text{alpha}\cdot x(t) - \text{beta}\cdot x(t)\cdot y(t), diff(y(t),t) = (\text{alpha}+1)\cdot y(t) - \text{gamma}\cdot\text{beta}\cdot x(t)\cdot y(t);$

$$armies := \frac{d}{dt} C_1(t) = 5\, C_1(t) - \frac{C_1(t)\, C_2(t)}{2500}, \frac{d}{dt} C_2(t) = 6\, C_2(t) - 0.0004600000000\, C_1(t)\, C_2(t) \quad \textbf{(24.15)}$$

$phaseportrait([armies], [x(t), y(t)], t=0..0.8, [[x(0)=4000, y(0)=3000], [x(0)=4000, y(0)=4500]],$
$x = 5000..30000, y = -5000..30000, dirgrid = [30, 30], stepsize = 0.01, linecolor = blue, arrows = MEDIUM);$



Stability Analysis for the Lorenz System (3d-nonlinear systems)

$restart : with(linalg) : N := 3 :$
$A := array(1 .. N, 1 .. N, [\ ]) :$
$eq[1] := \text{sigma}\cdot(x[2]-x[1]);$

$$eq_1 := \sigma\,(x_2 - x_1) \quad \textbf{(25.1)}$$

$eq[2] := r\cdot x[1]-x[2]-x[1]\cdot x[3];$

$$eq_2 := r\,x_1 - x_1\,x_3 - x_2 \quad \textbf{(25.2)}$$

$eq[3] := x[1]\cdot x[2] - b\cdot x[3];$

$$eq_3 := -b\,x_3 + x_1\,x_2 \quad \textbf{(25.3)}$$

$sol := solve(\{seq(eq[i], i=1 .. N)\}, \{seq(x[i], i=1 .. N)\});$

$$sol := \{x_1 = 0, x_2 = 0, x_3 = 0\}, \{x_1 = RootOf(\_Z^2 - r\,b + b), x_2 = RootOf(\_Z^2 - r\,b + b), x_3 = r - 1\} \quad \textbf{(25.4)}$$

removing RootOf **:**

$$\textit{removing RootOf} \quad \textbf{(25.5)}$$

$sols23 := allvalues(sol[2]);$

$$sols23 := \left\{x_1 = \sqrt{r\,b - b}, x_2 = \sqrt{r\,b - b}, x_3 = r - 1\right\}, \left\{x_1 = -\sqrt{r\,b - b}, x_2 = -\sqrt{r\,b - b}, x_3 = r - 1\right\} \quad \textbf{(25.6)}$$

$assign(sols23[1]);$
$x\_0 := x[1]; y\_0 := x[2]; z\_0 := x[3];$

$$x\_0 := \sqrt{r\,b - b}$$
$$y\_0 := \sqrt{r\,b - b}$$
$$z\_0 := r - 1 \quad \textbf{(25.7)}$$

for i from 1 to N do $x[i] := x[i] + u[i]$ od;

$$x_1 := \sqrt{b\,r - b} + u_1$$
$$x_2 := \sqrt{b\,r - b} + u_2$$

$$x_3 := r - 1 + u_3 \quad \textbf{(25.8)}$$

```
for i from 1 to N do
subs(u[2]=0, u[3]=0, eq[i]);
A[i, 1] := coeff(%, u[1]);
subs(u[1]=0, u[3]=0, eq[i]);
A[i, 2] := coeff(%, u[2]);
subs(u[1]=0, u[2]=0, eq[i]);
A[i, 3] := coeff(%, u[3]);
od:

print(A)
```

$$\begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b\,r - b} \\ \sqrt{b\,r - b} & \sqrt{b\,r - b} & -b \end{bmatrix} \quad \textbf{(25.9)}$$

```
cp := charpoly(A, lambda);
```

$$cp := b\,\lambda^2 + b\,\lambda\,r + b\,\lambda\,\sigma + 2\,b\,r\,\sigma + \lambda^3 + \lambda^2\,\sigma - 2\,\sigma\,b + \lambda^2 \quad \textbf{(25.10)}$$

```
lambda_eq := collect(cp, lambda);
```

$$lambda\_eq := \lambda^3 + (b + \sigma + 1)\,\lambda^2 + (b\,r + \sigma\,b)\,\lambda + 2\,b\,r\,\sigma - 2\,\sigma\,b \quad \textbf{(25.11)}$$

```
sigma := 10 : b := 8/3 : r := 28 :
num_lambda_eq := lambda_eq = 0;
```

$$num\_lambda\_eq := \lambda^3 + \frac{41}{3}\,\lambda^2 + \frac{304}{3}\,\lambda + 1440 = 0 \quad \textbf{(25.12)}$$

```
ans := evalf(solve(num_lambda_eq, lambda));
```

$$ans := -13.85457791, 0.093955624 - 10.19450522\,\mathrm{I}, 0.093955624 + 10.19450522\,\mathrm{I} \quad \textbf{(25.13)}$$

```
lambda[1] := ans[1]; lambda[2] := ans[2]; lambda[3] := ans[3];
```

$$\lambda_1 := -13.85457791$$

$$\lambda_2 := 0.093955624 - 10.19450522\,\mathrm{I}$$

$$\lambda_3 := 0.093955624 + 10.19450522\,\mathrm{I} \quad \textbf{(25.14)}$$

```
for i from 1 to 3 do
X[i] := evalf(subs(u[i]=0, x[i]));
od;
```

$$X_1 := 8.485281374$$

$$X_2 := 8.485281374$$

$$X_3 := 27. \quad (25.15)$$

Nonlinear Drag on a Falling Sphere and info of methods

restart :

de := *diff* (*v*(*t*), *t*) = *g* − *A*·*v*(*t*) − *B*·*v*(*t*)^2;

$$de := \frac{d}{dt} v(t) = g - A\,v(t) - B\,v(t)^2 \quad (26.1)$$

The following command line asks for **information on the methods used in the dsolve command** to obtain an analytic solution. The amount of information given, which will appear in the output of the dsolve command, depends on thelevel specified. The possible levels are 1, 2, 3, 4, or 5. Levels 2 and 3 provide general information on the technique or algorithm being used. More detailed information may sometimes be provided by specifying a higher level. In the following command line, the infolevel is taken to be 2.

infolevel[*dsolve*] := 2;

$$infolevel_{dsolve} := 2 \quad (26.2)$$

vel := *dsolve*({*de*, *v*(0) = 0}, *v*(*t*));

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful
```

$$vel := v(t) = -\frac{-\tanh\left(\frac{t\sqrt{A^2+4\,B\,g}}{2} + \operatorname{arctanh}\left(\frac{A}{\sqrt{A^2+4\,B\,g}}\right)\right)\sqrt{A^2+4\,B\,g} + A}{2\,B} \quad (26.3)$$

Isolation of V

restart :

eq := *int*(*x*, *x* = 0 ..*V*) = *g*·*t*

$$eq := \frac{V^2}{2} = g\,t \quad (27.1)$$

$isolate(eq, V)$

$$V = RootOf(\_Z^2 - 2\, g\, t) \quad \mathbf{(27.2)}$$

$allvalues(\%)$

$$V = \sqrt{2}\,\sqrt{g\,t},\ V = -\sqrt{2}\,\sqrt{g\,t} \quad \mathbf{(27.3)}$$

identify ODE with odeadvisor

$restart : with(DEtools, odeadvisor) :$

$ode := diff(y(x), x) + a \cdot y(x)^2 + \frac{y(x)}{x} + \frac{1}{a} = 0;$

$$ode := \frac{d}{dx}\, y(x) + a\, y(x)^2 + \frac{y(x)}{x} + \frac{1}{a} = 0 \quad \mathbf{(28.1)}$$

$odeadvisor(ode);$

$$[\_rational, \_Riccati] \quad \mathbf{(28.2)}$$

$dsolve(ode, y(x));$

$$y(x) = \frac{-\mathrm{I}\,\mathrm{BesselK}(1, \mathrm{I}\,x)\, c_1 + \mathrm{BesselJ}(1, x)}{a\,\left(c_1\, \mathrm{BesselK}(0, \mathrm{I}\,x) - \mathrm{BesselJ}(0, x)\right)} \quad \mathbf{(28.3)}$$

Series solution

$restart :$

$ODE := diff(x(t), t, t) + \sin(x(t)) = 0;$

$$ODE := \frac{d^2}{dt^2}\, x(t) + \sin(x(t)) = 0 \quad \mathbf{(29.1)}$$

$initvals := x(0) = \mathrm{Pi}/3, \mathrm{D}(x)(0) = 0;$

$$initvals := x(0) = \frac{\pi}{3}, \mathrm{D}(x)(0) = 0 \quad \mathbf{(29.2)}$$

$Order := 10 :$

$solution := dsolve(\{ODE, initvals\}, x(t), series);$

$$solution := x(t) = \frac{\pi}{3} - \frac{1}{4}\sqrt{3}\, t^2 + \frac{1}{96}\sqrt{3}\, t^4 + \frac{1}{720}\sqrt{3}\, t^6 - \frac{7}{46080}\sqrt{3}\, t^8 + \mathrm{O}(t^{10}) \quad \mathbf{(29.3)}$$

Generating Perturbation Equations and solution

restart : $N := 5$:
alias(epsilon = *ep*, tau = *t*) :#*creates Greek symbols*
$ODE := diff(X(t), t) + X(t) = -ep * X(t)^2$;

$$ODE := \frac{d}{d\tau} X(\tau) + X(\tau) = -\epsilon X(\tau)^2 \tag{30.1}$$

$X(t) := sum(x[i](t) \cdot ep^i, i = 0..N)$;

$$X := \tau \mapsto \sum_{i=0}^{N} x_i(\tau) \cdot \epsilon^i \tag{30.2}$$

$deqn := simplify(collect(ODE, ep), \{ep^{N+1} = 0\})$;

$$deqn := \left(\frac{d}{d\tau} x_5(\tau) + x_5(\tau)\right)\epsilon^5 + \left(\frac{d}{d\tau} x_4(\tau) + x_4(\tau)\right)\epsilon^4 + \left(\frac{d}{d\tau} x_3(\tau) + x_3(\tau)\right)\epsilon^3 + \left(\frac{d}{d\tau} x_2(\tau) + x_2(\tau)\right)\epsilon^2 + \left(\frac{d}{d\tau} x_1(\tau) + x_1(\tau)\right)\epsilon + \frac{d}{d\tau} x_0(\tau) + x_0(\tau) = -\epsilon \left(2 x_4(\tau) x_0(\tau) \epsilon^4 + 2 x_3(\tau) x_1(\tau) \epsilon^4 + x_2(\tau)^2 \epsilon^4 + 2 x_3(\tau) x_0(\tau) \epsilon^3 + 2 x_2(\tau) x_1(\tau) \epsilon^3 + 2 x_2(\tau) x_0(\tau) \epsilon^2 + x_1(\tau)^2 \epsilon^2 + 2 x_0(\tau) x_1(\tau) \epsilon + x_0(\tau)^2\right) \tag{30.3}$$

for *i* **from** 0 **to** *N* **do**
$eqn[i] := sort(coeff(lhs(deqn), ep, i)) = coeff(rhs(deqn), ep, i)$;
od;

$$eqn_0 := \frac{d}{d\tau} x_0(\tau) + x_0(\tau) = 0$$

$$eqn_1 := \frac{d}{d\tau} x_1(\tau) + x_1(\tau) = -x_0(\tau)^2$$

$$eqn_2 := \frac{d}{d\tau} x_2(\tau) + x_2(\tau) = -2 x_0(\tau) x_1(\tau)$$

$$eqn_3 := \frac{d}{d\tau} x_3(\tau) + x_3(\tau) = -2 x_0(\tau) x_2(\tau) - x_1(\tau)^2$$

$$eqn_4 := \frac{d}{d\tau} x_4(\tau) + x_4(\tau) = -2 x_0(\tau) x_3(\tau) - 2 x_1(\tau) x_2(\tau)$$

$$eqn_5 := \frac{d}{d\tau} x_5(\tau) + x_5(\tau) = -2 x_0(\tau) x_4(\tau) - 2 x_1(\tau) x_3(\tau) - x_2(\tau)^2 \tag{30.4}$$

Hard Spring Perturbation Solution

```
restart : N := 1 :
alias(epsilon = ep, tau = t) :#creates Greek symbols
ODE := W^2 diff(X(t), t, t) + omega[0]^2 X(t) + ep·X(t)^3 = 0;
```

$$ODE := W^2 \left(\frac{d^2}{d\tau^2} X(\tau)\right) + \omega_0^2 X(\tau) + \epsilon X(\tau)^3 = 0 \quad (31.1)$$

```
X(t) := sum(x[i](t)·ep^i, i = 0..N);
```

$$X := \tau \mapsto \sum_{i=0}^{N} x_i(\tau) \cdot \epsilon^i \quad (31.2)$$

```
W := omega[0] + sum(Omega[i]·ep^i, i = 1..N);
```

$$W := \Omega_1 \epsilon + \omega_0 \quad (31.3)$$

```
ODE2 := expand(ODE / omega[0]^2) :
deqn := simplify(collect(ODE2, ep), {ep^(N+1) = 0}) :
for i from 0 to N do
 eqn[i] := expand(coeff(lhs(deqn), ep, i)) =
 expand(coeff(rhs(deqn), ep, i));
 od;
```

$$eqn_0 := \frac{d^2}{d\tau^2} x_0(\tau) + x_0(\tau) = 0$$

$$eqn_1 := \frac{2\left(\frac{d^2}{d\tau^2} x_0(\tau)\right)\Omega_1}{\omega_0} + \frac{d^2}{d\tau^2} x_1(\tau) + x_1(\tau) + \frac{x_0(\tau)^3}{\omega_0^2} = 0 \quad (31.4)$$

```
sol[0] := dsolve({eqn[0], x[0](0) = A[0], D(x[0])(0) = 0}, x[0](t), method = laplace);
```

$$sol_0 := x_0(\tau) = A_0 \cos(\tau) \quad (31.5)$$

```
vel := rhs(sol[0]);
```

$$vel := A_0 \cos(\tau) \quad (31.6)$$

```
eq1 := subs(x[0](t) = vel, eqn[1]);
```

$$eq1 := \frac{2\left(\frac{\partial^2}{\partial\tau^2}\left(A_0 \cos(\tau)\right)\right)\Omega_1}{\omega_0} + \frac{d^2}{d\tau^2} x_1(\tau) + x_1(\tau) + \frac{A_0^3 \cos(\tau)^3}{\omega_0^2} = 0 \quad (31.7)$$

$eq1 := combine(lhs(eq1)) = 0:$

$eq[1] := collect(\%, [\cos, \text{omega}[0]^2]);$

$$eq_1 := \left(-\frac{2 A_0 \Omega_1}{\omega_0} + \frac{3 A_0^3}{4 \omega_0^2}\right)\cos(\tau) + \frac{A_0^3 \cos(3\tau)}{4 \omega_0^2} + x_1(\tau) + \frac{d^2}{d\tau^2} x_1(\tau) = 0 \quad \textbf{(31.8)}$$

$sol[1] := dsolve(\{eq[1], x[1](0) = 0, D(x[1])(0) = 0\}, x[1](t), method = laplace);$

$$sol_1 := x_1(\tau) = \frac{A_0 \tau \sin(\tau) \Omega_1}{\omega_0} + \frac{A_0^3 \left(\cos(\tau)^3 - 3\tau \sin(\tau) - \cos(\tau)\right)}{8 \omega_0^2} \quad \textbf{(31.9)}$$

$\text{Omega}[1] := solve(coeff(rhs(sol[1]), \sin(t)) = 0, \text{Omega}[1]);$

$$\Omega_1 := \frac{3 A_0^2}{8 \omega_0} \quad \textbf{(31.10)}$$

$subvel := subs(\sin(t) = 0, rhs(sol[1]));$

$$subvel := \frac{A_0^3 \left(\cos(\tau)^3 - \cos(\tau)\right)}{8 \omega_0^2} \quad \textbf{(31.11)}$$

$subvel := factor(subvel);$

$$subvel := \frac{A_0^3 \cos(\tau) \left(\cos(\tau) - 1\right)\left(\cos(\tau) + 1\right)}{8 \omega_0^2} \quad \textbf{(31.12)}$$

$X := rhs(sol[0]) + ep \cdot rhs(sol[1]);$

$$X := A_0 \cos(\tau) + \epsilon \left(\frac{3 A_0^3 \tau \sin(\tau)}{8 \omega_0^2} + \frac{A_0^3 \left(\cos(\tau)^3 - 3\tau \sin(\tau) - \cos(\tau)\right)}{8 \omega_0^2}\right) \quad \textbf{(31.13)}$$

$W := \text{omega}[0] + ep \cdot \text{Omega}[1];$

$$W := \frac{3 \epsilon A_0^2}{8 \omega_0} + \omega_0 \quad \textbf{(31.14)}$$

Euler Algorithm to solve numerically rabbits-foxes equations

```
restart : with(plots) :
t[0] := 0 : r[0] := 300 : f[0] := 150 : a := 0.01 : h := 0.02 : n := 1000 :
Digits := 10 : begin := time( ) :
for k from 0 to n do
```

$r[k+1] := r[k] + h\cdot(2\cdot r[k] - a\cdot r[k]\cdot f[k]);$
$f[k+1] := f[k] + h\cdot(-f[k] + a\cdot r[k]\cdot f[k]);$
$t[k+1] := t[k] + h;$
$pt[k] := [t[k], r[k], f[k]];$
od:

$cpu\_time := (time() - begin)\cdot seconds;$

$$cpu\_time := 0.156\ seconds \quad \textbf{(32.1)}$$

$pointplot3d([seq(pt[j], j = 0..n)], color = blue, axes = normal, labels = [t, r, f]);$