Generalization and analytic exploration of soliton solutions for nonlinear evolution equations via a novel symbolic approach in fluids and nonlinear sciences

Brij Mohan[*] and **Sachin Kumar**[†]

[*]Department of Mathematics, Hansraj College, University of Delhi, Delhi -110007, India

†Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi 110007, India

Abstract: In this work, we analyze the new generalized soliton solutions for the nonlinear partial differential equations with a novel symbolic bilinear technique. The proposed approach constructs the soliton solutions depending on the arbitrary parameters, which generalizes the soliton solutions with these additional parameters. Examining phase shifts and their dependence on the parameters influences how solitons collide, merge, or pass through each other, which is essential for the nonlinear analysis of solitons. Using the proposed technique, we examine the well-known (1+1)-dimensional Korteweg-de Vries (KdV) and (2+1)-dimensional Kadomtsev-Petviashvili (KP) equations with a comparative analysis of soliton solutions in the Hirota technique. We construct the generalized solitons solutions for both examined equations up to the third order, providing a better understanding of formed solitons with arbitrary parameter choices. The Cole-Hopf transformations are used to construct the bilinear form in the auxiliary function using Hirota's D-operators for both investigated KdV and KP equations. It discusses the phase shift depending on parameters and compares it to the phase shift in Hirota's soliton solutions. We utilize *Mathematica*, a computer algebra system, to obtain the generalized solitons and analyze the dynamic behavior of the obtained solutions by finding the values for the parameters and the relationships among them. Solitons are localized waves that appear in different fields of nonlinear sciences, such as oceanography, plasmas, fluid mechanics, water engineering, optical fibers, and other sciences.

Keywords: Analytical method; Generalized Solitons; Symbolic bilinear method; Cole-Hopf transformation; KdV equation; KP equation.

1 Introduction

Investigating solitons in nonlinear fields is a fascinating and pivotal area of research, offering deep insights into the behavior of solitary waves in diverse physical systems. Solitons, as well as stable and localized wave solutions, are essential to study in nonlinear sciences. The nonlinear Korteweg-de Vries (KdV) equation [1], Schrödinger equation [2], and Kadomtsev-Petviashvili (KP) equation [3] are renowned models that apprehend soliton dynamics in various contexts, such as optics, plasma, and fluid dynamics. Understanding solitons' formation, propagation, and interactions provides a deeper understanding of the relation among non-linearity, dispersion, and other relevant factors in nonlinear systems. The study of solitons enhances our theoretical and practical understanding of nonlinear phenomena. It holds practical implications, influencing technological advancements and developing novel applications in ocean engineering, plasma physics, telecommunication, and many nonlinear sciences.

[*] *brijmohan6414@gmail.com*

[†] *sachinambariya@gmail.com (Corresponding author)*

For decades, Hirota's bilinear method [4, 5] and simplified Hirota method [6, 7] have been used to obtain the soliton solutions of nonlinear partial differential equations (PDEs) in exact form, which uses the perturbation for the auxiliary function in dependent variable transformation. Now, we have numerous computer algebra systems or symbolic software available, such as *Maple, Matlab, Mathematica, Octave, Maxima, Scilab*, and some are freeware. Using such software, we can quickly investigate the highly computational work, such as finding exact solutions for nonlinear evolution equations. Researchers use several symbolic or computational methods to study the nonlinear PDE, such as the Hirota's bilinear approach for soliton solutions, symbolic computational approach for rogue wave solutions, simplified Hirota technique for soliton solutions, bilinear Bäcklund transformation for analytic solutions, Painelevé analysis for investigating integrability, and others. Hirota's bilinear method is popular among scientists and researchers for finding the exact soliton solutions, which provides a systematic approach to finding N-soliton solutions where N denotes the number of solitons. We can find several works in the literature to obtain the soliton solution using the Hirota bilinear approach. However, the solutions in Hirota's method do not provide the generalized solutions to study them with several arbitrary parameters.

This work introduces a novel symbolic bilinear technique for constructing generalized soliton solutions for nonlinear partial differential equations. The primary advantage of this method is its ability to incorporate arbitrary parameters into the solution, providing a more flexible and comprehensive framework than traditional methods such as the Hirota bilinear technique. We demonstrate that the proposed approach's solutions are exact and generalized to those obtained through Hirota's bilinear method. The investigation reveals the effectiveness and advantages of the presented approach by comparing the new soliton solutions with those of well-known KdV and KP equations. This flexibility allows for a detailed analysis of the behavior of soliton solutions under varying conditions, which can lead to new insights into the dynamics of these solutions. Moreover, our technique extends Hirota's soliton solutions by adding additional scaling parameters, thus offering a generalized approach that can be applied to a broader range of problems. It is particularly beneficial for examining complex systems where parameter variations can significantly impact the solution's properties. To ensure the robustness and validity of our proposed method, we performed a comparative analysis with soliton solutions obtained using Hirota's technique.
By applying our symbolic bilinear technique to the well-known (1+1)-dimensional Korteweg-de Vries (KdV) [1] and (2+1)-dimensional Kadomtsev-Petviashvili (KP) [3] equations, we verified that the generalized soliton solutions produced are consistent with established solutions, demonstrating the method's accuracy. Moreover, we analyzed the dynamical behavior of these solutions under various parameter values to confirm their stability and physical relevance. Using Mathematica allowed us to systematically explore the parameter space, ensuring that the solutions are mathematically valid and physically meaningful. Additionally, the ability to construct soliton solutions up to the third order highlighted the method's effectiveness, showcasing its potential for handling higher-order and more complex equations. These validations underscore the reliability and efficiency of our novel symbolic bilinear technique. Future studies will strengthen this validation by applying the method to additional nonlinear PDEs and comparing the results with experimental or observational data in relevant physical contexts.

Nonlinear partial differential equations (NLPDEs) represent a vast interdisciplinary domain within physics and applied mathematics. They serve as mathematical models for complex physical systems across diverse scientific fields. Investigating Nonlinear PDEs poses a formidable challenge due to the absence of universal techniques for their analysis. Therefore, each equation necessitates independent examination as a unique problem. However, certain situations may deserve broader approaches. Various techniques are employed to derive analytic and exact solutions, encompassing methodologies such as the Darboux transformation [8–11], simplified Hirota's technique [12, 13], Bäcklund transformation [14, 15], Bilinear Neural Network Method [16–19], Symbolic computation [20–27], Hirota's bilinear approach [28–31], Symmetry analysis [32, 33], Pfaffian technique [34, 35], and other methodologies.

The following section proposes the symbolic bilinear technique (SBT) and shows its different steps. In

Section 3, we study the (1+1)-D KdV equation utilizing the proposed technique and construct the generalized 1-, 2-, and 3-soliton solutions with their dynamics for distinct parameter values. Section 4 studies the (2+1)-D KP equation using SBT, obtains the generalized 1-, 2-, and 3-soliton solutions for distinct parameters, and shows the dynamics for the constructed solutions. In Section 5, we discuss the findings for the soliton solutions with the proposed technique, and at the end, we conclude the results and the work.

2 Symbolic Bilinear Technique (SBT)

Symbolic techniques [22–27] for solving nonlinear PDEs offer considerable advantages in mathematical physics and nonlinear sciences. One significant benefit is the ability to derive exact solutions, which provide deep insights into the underlying concepts and serve as benchmarks for validating complex phenomena. These techniques facilitate a deeper analytical understanding of system behavior, revealing relationships between variables and uncovering fundamental properties in soliton theory, plasma physics, and others. Symbolic methods often simplify complex nonlinear PDEs, transforming them into more tractable forms. Techniques like the Cole-Hopf transformation [28, 29] and Hirota's bilinear method [30, 31] can convert nonlinear equations into linear or bilinear forms, making them easier to solve. Additionally, symbolic techniques offer a systematic approach to finding higher-order solutions, such as solitons and rogue waves, which are crucial for understanding the dynamics of nonlinear systems. Symbolic techniques are versatile and applicable across various scientific disciplines, including fluid dynamics, plasma physics, and optical fibers, making them powerful tools for researchers.

Let us assume a nonlinear partial differential equation of (n+1)-dimensions with n spatial coordinates $\{x_1, x_2, x_3, \cdots, x_n\}$, and one temporal coordinate t as

$$S(u, u_t, u_{x_1}, u_{x_2}, u_{x_3}, u_{x_1x_1}, u_{x_1t}, u_{x_1x_2}, u_{x_1x_1x_1}, \cdots) = 0, \tag{1}$$

which contains partial derivatives with independent variables $\{x_1, x_2, x_3, \cdots, x_n, t\}$ to dependent variable function u.

First we transform the equation (1) by constructing a Cole-Hopf transformation [28–31]

$$u(x_1, x_2, x_3, \cdots, x_n, t) = R(\ln f)_{x_i^m}, \tag{2}$$

where R is a nonzero real constant and $f(x_1, x_2, x_3, \cdots, x_n, t)$ is an auxiliary function, m is the order of i^{th} independent variable x_i, obtained by balancing between nonlinear and higher-order terms in PDE for x_i. The equation (1) is changed by Cole-Hopf transformation to a bilinear equation in auxiliary function f as

$$T(f, f_t, f_{x_1}, f_{x_2}, f_{x_3}, f_{x_1x_1}, f_{x_1t}, f_{x_1x_2}, f_{x_1x_1x_1}, \cdots) = 0, \tag{3}$$

which can be represented in the Hirota's bilinear form with D-operators as

$$H(D_t, D_{x_1}, D_{x_2}, D_{x_3}, D_{x_1}^2, D_{x_1}D_{x_2}, D_{x_2}^2, D_{x_2}D_{x_3}, \cdots)f.f = 0. \tag{4}$$

For obtaining the N-soliton solution for equation (4), we express the auxiliary function f symbolically as

$$f = \sum_{k_{\{i=1,2,\ldots,N\}}=0,1}^{\{2^N\}} a_{k_1,k_2,k_3,\ldots,k_N} \mathrm{e}^{k_1\xi_1 + k_2\xi_2 + k_3\xi_3 + \cdots + k_N\xi_N}, \tag{5}$$

where $k_i = 0, 1$ are the binary choices for $1 \le i \le N$, 2^N represents the number of terms, $a_{k_1,k_2,k_3,\ldots,k_N}$ are the real non-zero parameters to be determined, and ξ_i are the phases for the equation (4)

For $N = 1$, we get $k_1 = 0, 1$, therefore

$$f = \sum_{k_1=0,1}^{\{2\}} a_{k_1} \mathrm{e}^{k_1\xi_1} = a_0 + a_1 \mathrm{e}^{\xi_1}.$$

For $N = 2$, we have $k_1, k_2 = 0, 1$, therefore

$$f = \sum_{k_1,k_2=0,1}^{\{4\}} a_{k_1,k_2} \mathrm{e}^{k_1\xi_1+k_2\xi_2} = a_{0,0} + a_{1,0}\mathrm{e}^{\xi_1} + a_{0,1}\mathrm{e}^{\xi_2} + a_{1,1}\mathrm{e}^{\xi_1+\xi_2}.$$

For $N = 3$, we get $k_1, k_2, k_3 = 0, 1$ therefore

$$\begin{aligned} f &= \sum_{k_1,k_2,k_3=0,1}^{\{8\}} a_{k_1,k_2,k_3} \mathrm{e}^{k_1\xi_1+k_2\xi_2+k_3\xi_3} \\ &= a_{0,0,0} + a_{1,0,0}\mathrm{e}^{\xi_1} + a_{0,1,0}\mathrm{e}^{\xi_2} + a_{0,0,1}\mathrm{e}^{\xi_3} + a_{1,1,0}\mathrm{e}^{\xi_1+\xi_2} + a_{1,0,1}\mathrm{e}^{\xi_1+\xi_3} + a_{0,1,1}\mathrm{e}^{\xi_2+\xi_3} + a_{1,1,1}\mathrm{e}^{\xi_1+\xi_2+\xi_3}. \end{aligned}$$

Thus, the auxiliary functions f provide the N-soliton solutions of the bilinear equation (4) for different values of $N = 1, 2, 3$, that are solutions to the equation (1). The symbolic bilinear technique obtains the generalized N-soliton solutions depending on arbitrary parameters and observes that the Hirota's N-soliton solutions [1] using bilinear method [29–31] are as one case for the obtained solution by this symbolic approach.
For $a_0 = a_1 = 1$, $a_{0,0} = a_{1,0} = a_{0,1} = 1$, and $a_{0,0,0} = a_{1,0,0} = a_{0,1,0} = a_{0,0,1} = 1$, the above auxiliary functions generate Hirota's [1] 1-soliton, 2-soliton, and 3-soliton solutions, respectively. Thus, the solutions by this approach have the opportunities to observe and study the behavior of solitons with different values of these additional real parameters $a_{k_1,k_2,k_3,\ldots,k_N}$ along with the constants presents in the phase variables for the studied equations as discussed in the following section.

3 (1+1)-dimensional KdV equation

The nonlinear KdV equation [1] is an evolution equation that describes the evolution of one-dimensional, weakly nonlinear, and long waves. It was first introduced in the field of hydrodynamics to model the behavior of shallow water waves. The KdV equation is

$$u_t + 6uu_x + u_{xxx} = 0, \tag{6}$$

where u is the dependent variable that represents the wave amplitude, x and t are the spatial coordinate and time, respectively. The equation 6 is particularly notable for its soliton solutions, which are solitary wave solutions that maintain their shape and speed during propagation. These solitons arise due to a balance among nonlinear and dispersive terms in the equation. The nonlinear KdV equation has applications in several domains, including plasma physics, fluid dynamics, and nonlinear optics, making it a fundamental model for studying wave phenomena.
Let us consider the phase variable ξ_i in the KdV equation (6) as

$$\xi_i = \mu_i x - d_i t, \tag{7}$$

with $\mu_i : i = 1, 2, \ldots$, as constant parameters and d_i as the dispersion coefficients. On putting $u = \mathrm{e}^{\xi_i}$ in linear terms of Eq. (6), we get the dispersion $d_i = \mu_i^3$.
Considering the transformation

$$u(x,t) = R(\ln f)_{xx}, \tag{8}$$

and putting it with $f(x,t) = 1+\mathrm{e}^{\xi_1}$ in Eq. (6). On solving, we get R as 2. Thus, the equation (8) transforms the equation (6) into a bilinear equation in f as

$$ff_{xt} - f_x f_t + 3f_{xx}^2 - 4f_x f_{xxx} + ff_{xxxx} = 0. \tag{9}$$

Using the bilinear differential operators $D_i : i = X, T$ defined by Hirota [1] as

$$D_X^{R_1} D_T^{R_2} UV = \left(\frac{\partial}{\partial_X} - \frac{\partial}{\partial_{X'}}\right)^{R_1} \left(\frac{\partial}{\partial_T} - \frac{\partial}{\partial_{T'}}\right)^{R_2} U(X,T)V(X',T')|_{X=X',T=T'},$$

with X' and T' as the formal variables and $R_j : j = 1, 2$ as positive integers. Thus, the bilinear equation (9) can be written in the Hirota's bilinear form with bilinear differentials D as

$$[D_x D_t + D_x^4]f.f = 0. \tag{10}$$

For one soliton solution, we take the function f in Eq. (10) by having $N = 1$ in the equation (5) as

$$f_1 = f(x,t) = \sum_{k_1=0,1}^{\{2\}} a_{k_1} \mathrm{e}^{k_1 \xi_1} = a_0 + a_1 \mathrm{e}^{\xi_1} = a_0 + a_1 \mathrm{e}^{\mu_1 x - d_1 t}, \tag{11}$$

which satisfies the bilinear equation (10) identically with arbitrary choices of a_0 and a_1. By substituting the equations (11) into (8), we get a generalized solution for one-soliton as

$$u_1(x,t) = u = \frac{2a_0 a_1 \mu_1^2 e^{\mu_1^3 t + \mu_1 x}}{\left(a_0 e^{\mu_1^3 t} + a_1 e^{\mu_1 x}\right)^2}, \tag{12}$$

that dependents on the arbitrary parameters a_0 and a_1. Therefore, we analyze the behavior and dynamics of this 1-soliton solution for distinct values of these parameters. As, we have discussed that for $a_0 = a_1 = 1$ the solution will give the Hirota's 1-soliton solution [1], thus, the obtained solution is a generalized 1-soliton solution with arbitrary choice of non-zero parameters, and the dynamics are shown in figure 1.

With $N = 2$, the equation (5) gives auxiliary function f as

$$f_2 = f = \sum_{k_1,k_2=0,1}^{\{4\}} a_{k_1,k_2} \mathrm{e}^{k_1\xi_1 + k_2\xi_2} = a_{0,0} + a_{1,0}\mathrm{e}^{\xi_1} + a_{0,1}\mathrm{e}^{\xi_2} + a_{1,1}\mathrm{e}^{\xi_1+\xi_2}. \tag{13}$$

By substituting the Eq. (13) into bilinear Eq. (10), and equating the coefficients of distinct expressions in power of exponential functions to zero, we get

$$a_{1,1} = \frac{(\mu_1 - \mu_2)^2 a_{0,1} a_{1,0}}{(\mu_1 + \mu_2)^2 a_{0,0}}. \tag{14}$$

On substituting Eqs (13) into (8), give a two-soliton solution of equation (6) as

$$u_2(x,t) = u = 2(\ln f_2)_{xx}, \tag{15}$$

that dependents on the arbitrary parameters $a_{0,0}, a_{0,1}$ and $a_{1,0}$. Therefore, we study the dynamical behavior of this 2-soliton solution for distinct values of these arbitrary non-zero parameters. For $a_{0,0} = a_{0,1} = a_{1,0} = 1$ the solution (15) will represent a Hirota's 2-soliton solution [1] and the equation (14) shows the phase shift

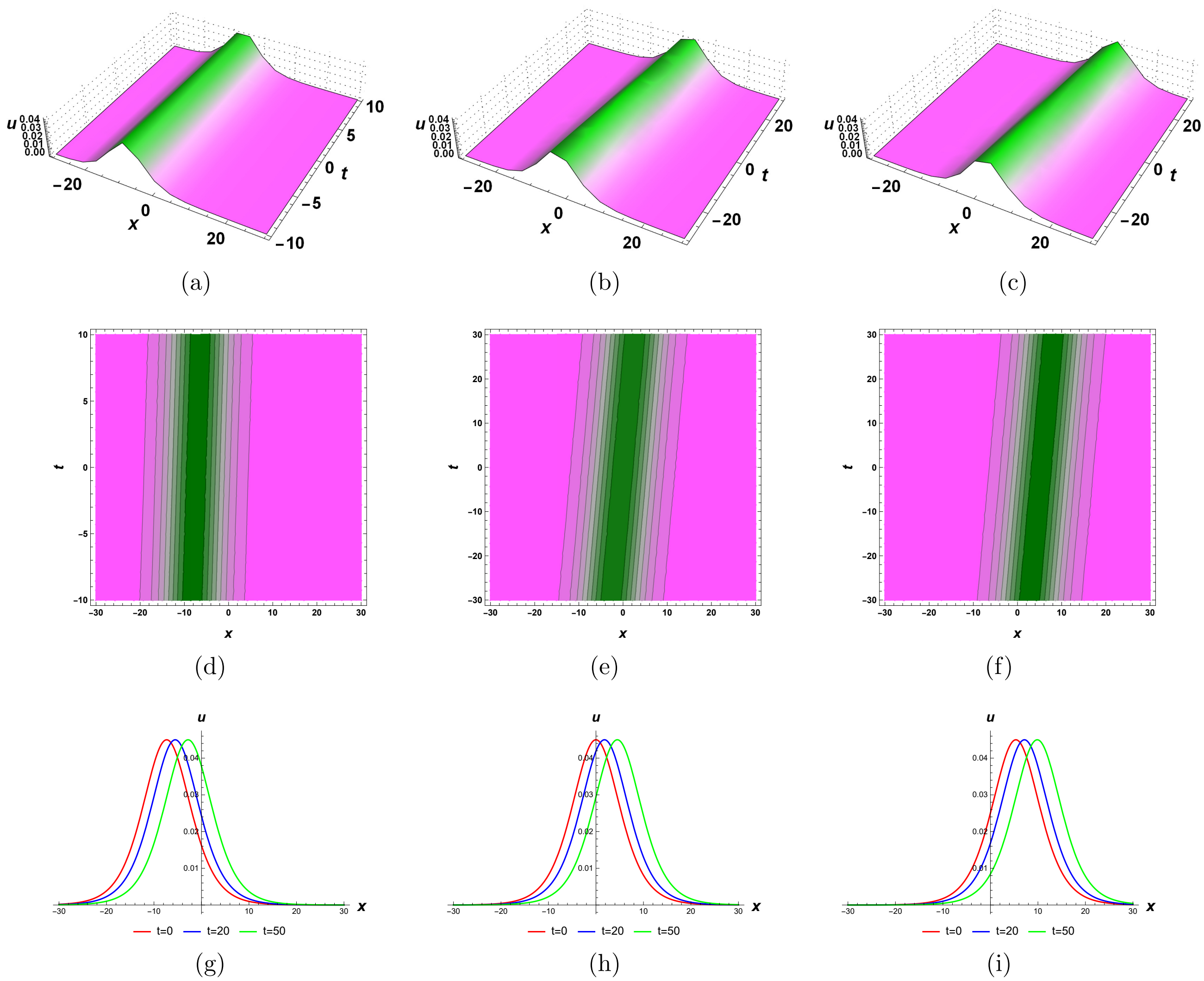


Figure 1: Dynamical profiles of 1-soliton solitons for (12) with $\mu_1 = 0.3$ and **(a)** $a_0 = 0.1, a_1 = 0.9$; **(b)** $a_0 = 1, a_1 = 1$ **(c)** $a_0 = -0.5, a_1 = -0.1$; **(d)-(f)** and **(g)-(i)** depicts the contour and 2D plots for **(a)-(c)**, respectively.

in Hirota's bilinear method. Thus, the obtained solution is a generalized 2-soliton solution with these parameters, and the figure 2 shows the dynamics for this solution.

For $N = 3$ in Eq. (5), we consider the auxiliary function f as

$$\begin{aligned} f_3 = f &= \sum_{k_1,k_2,k_3=0,1}^{\{8\}} a_{k_1,k_2,k_3} \mathrm{e}^{k_1\xi_1+k_2\xi_2+k_3\xi_3} \qquad (16) \\ &= a_{0,0,0} + a_{1,0,0}\mathrm{e}^{\xi_1} + a_{0,1,0}\mathrm{e}^{\xi_2} + a_{0,0,1}\mathrm{e}^{\xi_3} + a_{1,1,0}\mathrm{e}^{\xi_1+\xi_2} + a_{1,0,1}\mathrm{e}^{\xi_1+\xi_3} + a_{0,1,1}\mathrm{e}^{\xi_2+\xi_3} + a_{1,1,1}\mathrm{e}^{\xi_1+\xi_2+\xi_3}. \end{aligned}$$

On substituting the Eq. (16) into the bilinear Eq. (10), and equating the coefficients of distinct expressions in power of exponential functions to zero, we get

$$a_{1,1,0} = \frac{(\mu_1 - \mu_2)^2 a_{0,1,0} a_{1,0,0}}{(\mu_1 + \mu_2)^2 a_{0,0,0}}, \quad a_{0,1,1} = \frac{(\mu_2 - \mu_3)^2 a_{0,0,1} a_{0,1,0}}{(\mu_2 + \mu_3)^2 a_{0,0,0}}, \quad a_{1,0,1} = \frac{(\mu_1 - \mu_3)^2 a_{0,0,1} a_{1,0,0}}{(\mu_1 + \mu_3)^2 a_{0,0,0}},$$

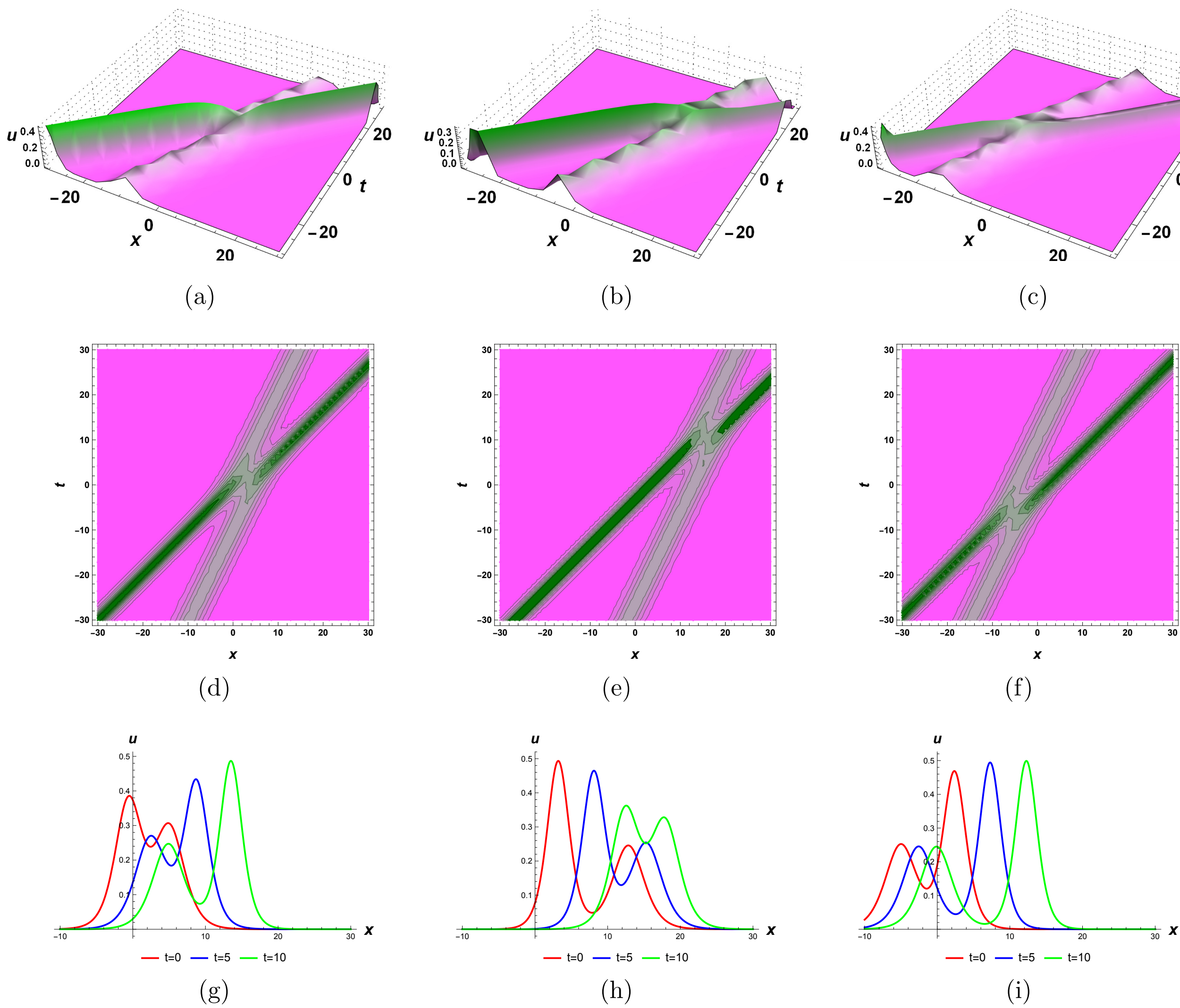


Figure 2: Dynamical profiles of 2-soliton solitons for (15) with $\mu_1 = 0.7, \mu_2 = 1$ and **(a)** $a_{0,0} = 1, a_{1,0} = 1, a_{0,1} = 1$ **(b)** $a_{0,0} = 25, a_{1,0} = 0.5, a_{0,1} = 1$ **(c)** $a_{0,0} = 0.3, a_{1,0} = 10, a_{0,1} = 1$; **(d)-(f)** and **(g)-(i)** depicts the contour and 2D plots for **(a)-(c)**, respectively.

$$a_{1,1,1} = \frac{(\mu_1 - \mu_2)^2 (\mu_1 - \mu_3)^2 (\mu_2 - \mu_3)^2 a_{0,0,1} a_{0,1,0} a_{1,0,0}}{(\mu_1 + \mu_2)^2 (\mu_1 + \mu_3)^2 (\mu_2 + \mu_3)^2 a_{0,0,0}^2},$$

which shows the relation as

$$a_{1,1,1} = (a_{1,1,0} \times a_{0,1,1} \times a_{1,0,1}) \frac{a_{0,0,0}}{a_{1,0,0} a_{0,1,0} a_{0,0,1}}, \tag{17}$$

where $a_{0,0,0}, a_{1,0,0}, a_{0,1,0}$ and $a_{0,0,1}$ are arbitrary constants. Thus the equation (16) satisfies the equation (10) as a solution with the above parameters. By substituting the equation (16) into (8), we establish a 3-soliton solution as

$$u_3(x,t) = u = 2(\ln f_3)_{xx}, \tag{18}$$

that dependents on the arbitrary parameters $a_{0,0,0}, a_{1,0,0}, a_{0,1,0}$ and $a_{0,0,1}$. Therefore, we study the behavior with dynamics of this 3-soliton solution for distinct values of these non-zero parameters. For $a_{0,0,0} = a_{1,0,0} = a_{0,1,0} = a_{0,0,1} = 1$ the solution (18) will represent a Hirota's 3-soliton solution [1] and the equation

(17) satisfies the disperssion relation for the parameters as in Hirota's bilinear method, thus, the obtained solution is a generalized three-soliton solution with arbitrary parameters, and the dynamics are shown in figure 3.

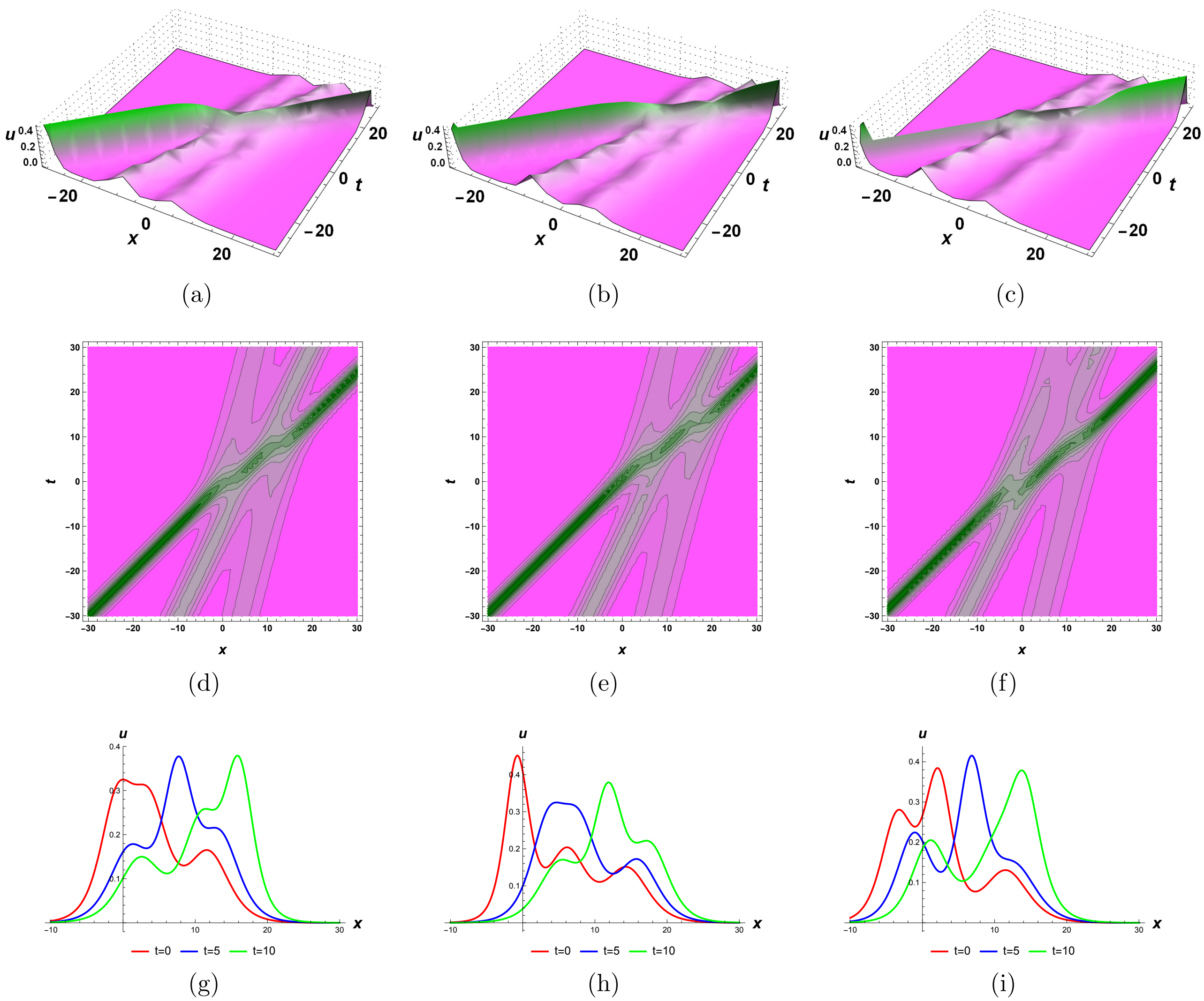


Figure 3: Dynamical profiles of 3-soliton solitons for (15) with $\mu_1 = 0.7, \mu_2 = 1, \mu_3 = 0.5$ and **(a)** $a_{0,0,0} = a_{1,0,0} = a_{0,1,0} = a_{0,0,1} = 1$ **(b)** $a_{0,0,0} = 4, a_{1,0,0} = 1, a_{0,1,0} = 8, a_{0,0,1} = 1$ **(c)** $a_{0,0,0} = 1, a_{1,0,0} = 10, a_{0,1,0} = 5, a_{0,0,1} = 1$; **(d)-(f)** and **(g)-(i)** depicts the contour and 2D plots for **(a)-(c)**, respectively.

4 (2+1)-dimensional KP equation

The nonlinear KP equation [3] significantly extends the Korteweg-de Vries (KdV) equation, specifically developed to describe two-dimensional weakly nonlinear and dispersive waves. Its mathematical form is

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0, \tag{19}$$

where u represents the wave amplitude, x, y and t are spatial and time coordinate, respectively. The nonlinear KP equation is essential to studying solitons and integrable systems exhibiting rich and complex

behaviors of wave interactions. Similar to the KdV equation, the KP equation supports solitons, but its two-dimensional nature allows for more elaborate structures, such as two-soliton solutions that interact in a nontrivial manner. The nonlinear KP equation finds applications in diverse fields, such as plasma, fluid mechanics, and oceanography, contributing to our understanding of nonlinear wave phenomena in multiple dimensions.
Let us consider the phase variable ξ_i in the KP equation (19) as

$$\xi_i = \mu_i x + \nu_i y - d_i t, \tag{20}$$

with $\mu_i, \nu_i : i = 1, 2, ...,$ as constant parameters and d_i as the dispersion coefficients. On putting $u = \mathrm{e}^{\xi_i}$ in linear terms of Eq. (19), we get the dispersion $d_i = \frac{\mu_i^4 - \nu_i^2}{\mu_i}$.
Considering the transformation

$$u(x, y, t) = R(\ln f)_{xx}, \tag{21}$$

and putting it with $f(x, y, t) = 1 + \mathrm{e}^{\xi_1}$ in Eq. (19). On solving, we get R as 2. Thus, the equation (21) transforms the equation (19) into a bilinear equation in f as

$$ff_{xt} - f_x f_t + 3f_{xx}^2 - 4f_x f_{xxx} + ff_{xxxx} - ff_{yy} + f_y^2 = 0. \tag{22}$$

Using the bilinear operators $D_i : i = X, Y, T$ defined by Hirota [1] as

$$D_X^{R_1} D_Y^{R_2} D_T^{R_3} UV = \left(\frac{\partial}{\partial_X} - \frac{\partial}{\partial_{X'}}\right)^{R_1} \left(\frac{\partial}{\partial_Y} - \frac{\partial}{\partial_{Y'}}\right)^{R_2} \left(\frac{\partial}{\partial_T} - \frac{\partial}{\partial_{T'}}\right)^{R_3} U(X, Y, T)V(X', Y', T')|_{X=X', Y=Y', T=T'},$$

with X', Y' and T' as the formal variables and $R_j : j = 1, 2, 3$ as positive integers. Thus, the bilinear equation (22) can be written in the Hirota's bilinear form with bilinear differentials D as

$$[D_x D_t + D_x^4 - D_y^2]f.f = 0. \tag{23}$$

For $N = 1$ in the equation (5), we take the function f in Eq. (23) as

$$f_1 = f(x, y, t) = \sum_{k_1=0,1}^{\{2\}} a_{k_1} \mathrm{e}^{k_1 \xi_1} = a_0 + a_1 \mathrm{e}^{\xi_1} = a_0 + a_1 \mathrm{e}^{\mu_1 x + \nu_1 y - d_1 t}, \tag{24}$$

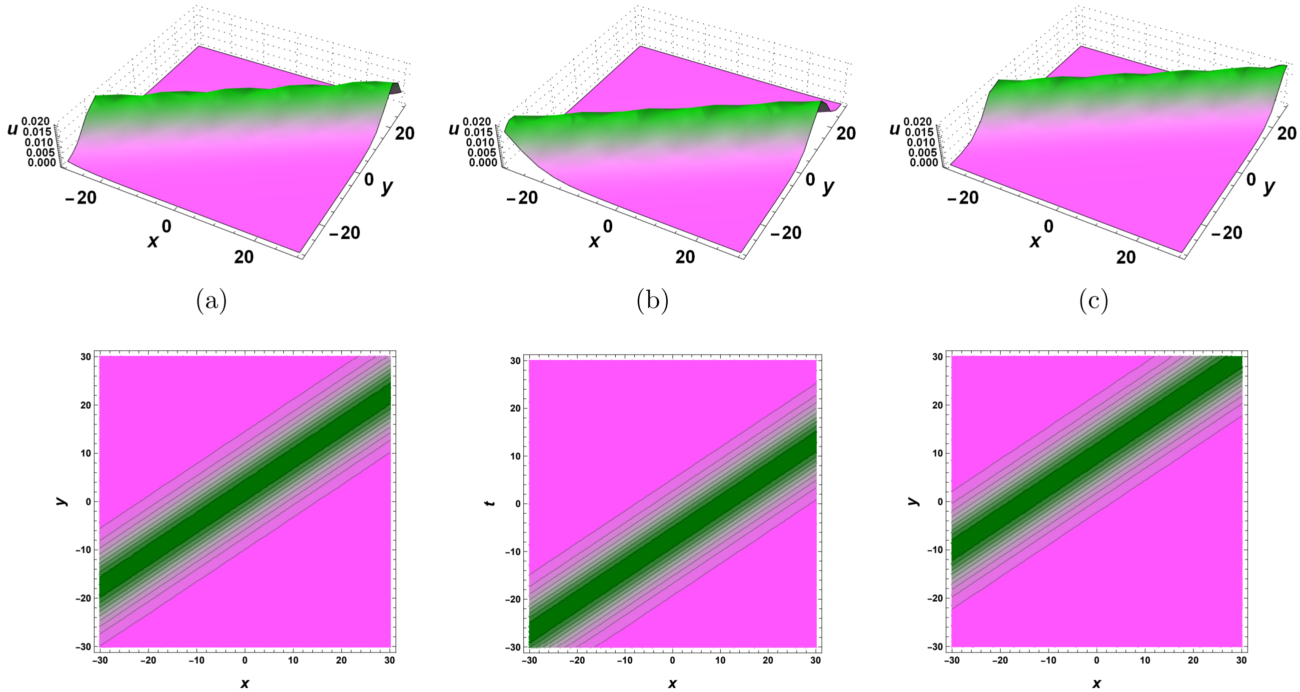
which satisfies the bilinear equation (23) identically with arbitrary choices of a_0 and a_1. By substituting the equations (24) into (21), we get one-soliton solution as

$$u_1(x, y, t) = u = \frac{2a_0 a_1 \mu_1^2 e^{\frac{\nu_1^2 t}{\mu_1} + \mu_1^3 t + \mu_1 x + \nu_1 y}}{\left(a_1 e^{\frac{\nu_1^2 t}{\mu_1} + \mu_1 x + \nu_1 y} + a_0 e^{\mu_1^3 t}\right)^2}, \tag{25}$$

that dependents on the parameters a_0 and a_1. Therefore, we observe the behavior and the dynamics of this 1-soliton solution for distinct values of these non-zero parameters. For $a_0 = a_1 = 1$ the solution will represent the Hirota's 1-soliton solution [3], thus, the obtained solution is a generalized 1-soliton solutions with these non-zero parameters, and the dynamics are shown in figure 4.

Having $N = 2$ in the equation (5), we take auxiliary function f as

$$f_2 = f = \sum_{k_1, k_2=0,1}^{\{4\}} a_{k_1, k_2} \mathrm{e}^{k_1 \xi_1 + k_2 \xi_2} = a_{0,0} + a_{1,0} \mathrm{e}^{\xi_1} + a_{0,1} \mathrm{e}^{\xi_2} + a_{1,1} \mathrm{e}^{\xi_1 + \xi_2}. \tag{26}$$



(a) (b) (c)

(d) (e) (f)

Figure 4: Dynamical profiles of 1-soliton solutions for (25) with $\mu_1 = 0.2, \nu_1 = -0.3, t = 0$ and **(a)** $a_0 = 0.1, a_1 = 0.2$; **(b)** $a_0 = 0.8, a_1 = 0.11$ **(c)** $a_0 = 1, a_1 = 20$; **(d)-(f)** depicts the contour plots for **(a)-(c)**, respectively.

By substituting the Eq. (26) into the bilinear Eq. (23), and equating the coefficients of distinct expressions in power of exponential functions to zero, we get

$$a_{1,1} = \frac{\left(3\mu_1^2\mu_2^2(\mu_1 - \mu_2)^2 + (\mu_1\nu_2 - \mu_2\nu_1)^2\right) a_{0,1}a_{1,0}}{\left(3\mu_1^2\mu_2^2(\mu_1 + \mu_2)^2 + (\mu_1\nu_2 - \mu_2\nu_1)^2\right) a_{0,0}}. \tag{27}$$

On substituting Eq. (26) into the equation (21), give a two-soliton solution of equation (19) as

$$u_2(x,t) = u = 2(\ln f_2)_{xx}, \tag{28}$$

that dependents on the arbitrary parameters $a_{0,0}, a_{0,1}$ and $a_{1,0}$. Therefore, we study the dynamics of this 2-soliton solution for distinct values of these non-zero parameters. For $a_{0,0} = a_{0,1} = a_{1,0} = 1$, the solution (28) will represent a Hirota's 2-soliton solution [3] and the equation (27) shows the relation for the phase shift as in Hirota's bilinear method, thus, the obtained solution is a generalized 2-soliton solution with these arbitrary parameters, and the figure 5 shows the dynamics of the solution.

For $N = 3$ in the equation (5), we consider the function f as

$$\begin{aligned} f_3 = f &= \sum_{k_1,k_2,k_3=0,1}^{\{8\}} a_{k_1,k_2,k_3} e^{k_1\xi_1 + k_2\xi_2 + k_3\xi_3} \qquad (29) \\ &= a_{0,0,0} + a_{1,0,0}e^{\xi_1} + a_{0,1,0}e^{\xi_2} + a_{0,0,1}e^{\xi_3} + a_{1,1,0}e^{\xi_1+\xi_2} + a_{1,0,1}e^{\xi_1+\xi_3} + a_{0,1,1}e^{\xi_2+\xi_3} + a_{1,1,1}e^{\xi_1+\xi_2+\xi_3}. \end{aligned}$$

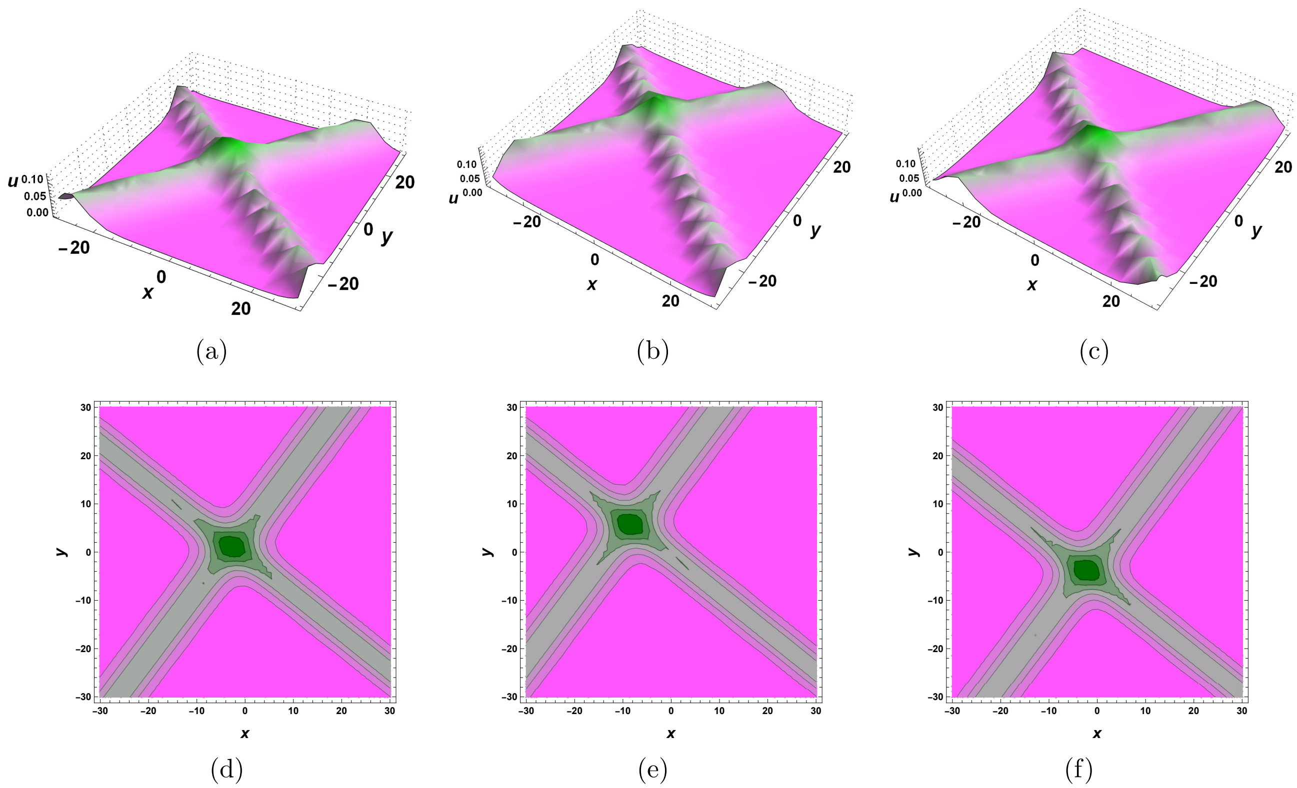


Figure 5: Dynamical profiles of 2-soliton solutions for (15) with $\mu_1 = \mu_2 = 0.4, \nu_1 = 0.5, \nu_2 = -0.3, t = 0$ and **(a)** $a_{0,0} = 0.1, a_{1,0} = 0.2, a_{0,1} = 0.5$ **(b)** $a_{0,0} = 0.1, a_{1,0} = 0.2, a_{0,1} = 20$ **(c)** $a_{0,0} = 1, a_{1,0} = 20, a_{0,1} = 1$; **(d)-(f)** depicts the contour plots for **(a)-(c)**, respectively.

On substituting the equation (29) into the bilinear Eq. (23), and equating the coefficients of distinct expressions in power of exponential functions to zero, we get

$$
\begin{aligned}
a_{1,1,0} &= \frac{\left(3\mu_1^2\mu_2^2(\mu_1-\mu_2)^2 + (\mu_1\nu_2-\mu_2\nu_1)^2\right) a_{1,0,0}a_{0,1,0}}{\left(3\mu_1^2\mu_2^2(\mu_1+\mu_2)^2 + (\mu_1\nu_2-\mu_2\nu_1)^2\right) a_{0,0,0}},\\
a_{0,1,1} &= \frac{\left(3\mu_2^2\mu_3^2(\mu_2-\mu_3)^2 + (\mu_2\nu_3-\mu_3\nu_2)^2\right) a_{0,1,0}a_{0,0,1}}{\left(3\mu_2^2\mu_3^2(\mu_2+\mu_3)^2 + (\mu_2\nu_3-\mu_3\nu_2)^2\right) a_{0,0,0}},\\
a_{1,0,1} &= \frac{\left(3\mu_1^2\mu_3^2(\mu_1-\mu_3)^2 + (\mu_1\nu_3-\mu_3\nu_1)^2\right) a_{1,0,0}a_{0,0,1}}{\left(3\mu_1^2\mu_3^2(\mu_1+\mu_3)^2 + (\mu_1\nu_3-\mu_3\nu_1)^2\right) a_{0,0,0}},\\
a_{1,1,1} &= (a_{1,1,0}\times a_{0,1,1}\times a_{1,0,1})\frac{a_{0,0,0}}{a_{1,0,0}a_{0,1,0}a_{0,0,1}},
\end{aligned}
\tag{30}
$$

where $a_{0,0,0}, a_{1,0,0}, a_{0,1,0}$ and $a_{0,0,1}$ are arbitrary constants. Thus the equation (29) satisfies the equation (23) as a solution with the arbitrary non-zero parameters. By substituting the equation (29) into (21), we establish a 3-soliton solution as

$$u_3(x,y,t) = u = 2(\ln f_3)_{xx}, \tag{31}$$

that depends on the arbitrary parameters $a_{0,0,0}, a_{1,0,0}, a_{0,1,0}$ and $a_{0,0,1}$. Therefore, we study the dynamics of this 3-soliton solution for distinct values of these non-zero parameters. For $a_{0,0,0} = a_{1,0,0} = a_{0,1,0} = a_{0,0,1} = 1$ the solution will represent Hirota's 3-soliton solution [3] and the equation (30) satisfies the dispersion relation for the parameters as in Hirota's bilinear technique, thus, the obtained solution is a generalized three-soliton solution with these non-zero parameters, and the dynamics are shown in figure 6.

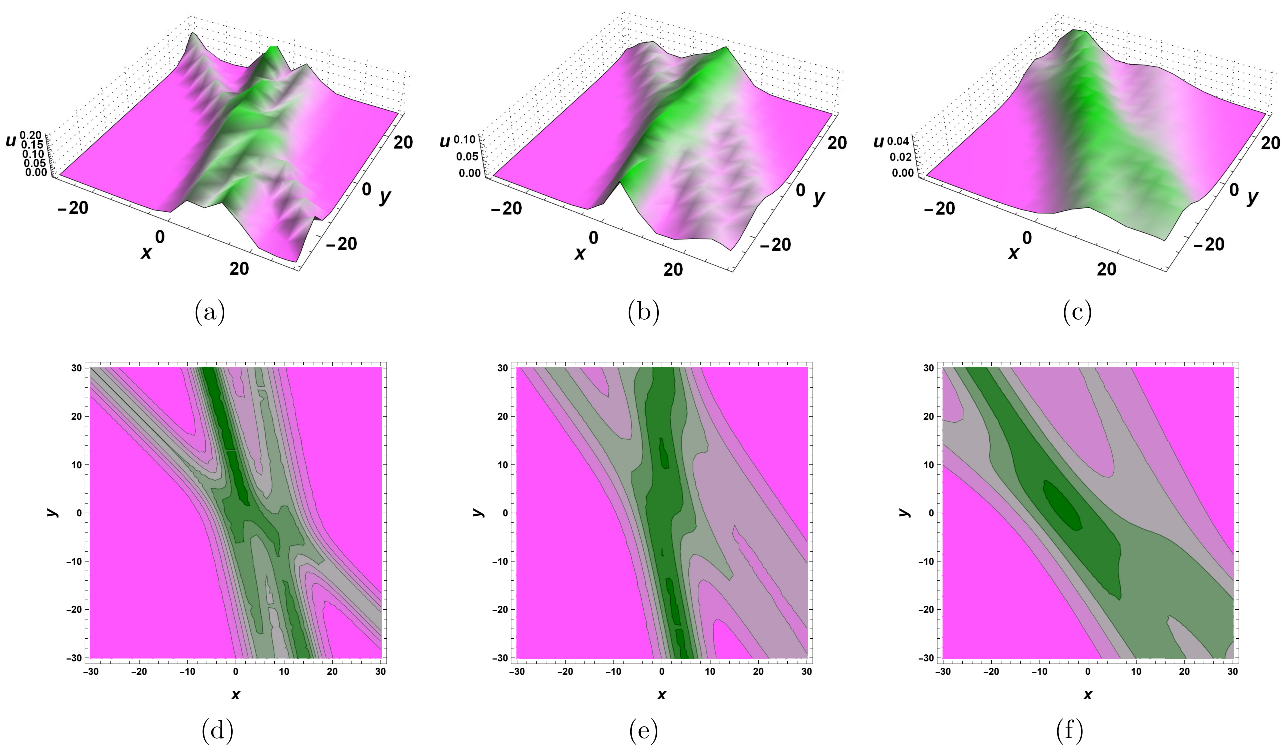


Figure 6: Dynamical profiles of 3-soliton solutions for (31) with $t = 0$ and **(a)** $a_{0,0,0} = a_{1,0,0} = a_{0,1,0} = a_{0,0,1} = 1, \mu_1 = \mu_3 = 0.5, \mu_2 = 0.7, \nu_1 = 0.5, \nu_2 = 0.2, \nu_3 = 0.1$ **(b)** $a_{0,0,0} = 0.5, a_{1,0,0} = 0.3, a_{0,1,0} = 0.5, a_{0,0,1} = 1, \mu_1 = 0.4, \mu_2 = 0.3, \mu_3 = 0.5, \nu_1 = 0.3, \nu_2 = 0.2, \nu_3 = 0.1$ **(c)** $a_{0,0,0} = 0.1, a_{1,0,0} = 0.3, a_{0,1,0} = 0.5, a_{0,0,1} = 0.1, \mu_1 = 0.2, \mu_2 = 0.3, \mu_3 = 0.2, \nu_1 = 0.3, \nu_2 = 0.2, \nu_3 = 0.1$; **(d)-(f)** depicts the contour plots for **(a)-(c)**, respectively.

5 Results and discussion

This work has analyzed the newly constructed generalized soliton solutions concerning arbitrary parameters utilizing the proposed symbolic bilinear technique. Our analysis includes the examination of phase shifts and their dependence on the parameters, which is essential for characterizing the interaction of solitons in physical systems [1–4]. Additionally, the generalized solitons produced by our approach provide a more realistic representation of physical processes due to the inclusion of arbitrary parameters. This approach verifies that the solutions obtained in Hirota's bilinear approach [1, 3] for the studied equations are a case for the established generalized solutions. We have shown a comparative analysis of the existing solutions for well-known KdV and KP equations using Hirota's bilinear method and the solutions using our proposed approach. Researchers and investigators can apply this technique to the other equations to more broadly understand the behavior and physical appearance of the solutions for a nonlinear system [5–9].
The analysis of generalized soliton solutions using the symbolic bilinear technique reveals several critical aspects of their physical significance. These include the flexibility introduced by arbitrary parameters, the importance of phase shifts in soliton interactions, and the validation of the proposed approach against established methods. Generalized soliton solutions that arbitrarily provide additional parameters make the

description of physical processes more universal and complete, giving a better understanding of their underlying dynamics. The study of phase shifts as a function of these parameters is particularly relevant for understanding soliton interactions in physical systems. Otherwise, phase shifts can play a crucial role in influencing the collisions of solitons to merge like this or cross each other and are, therefore, essential for the nonlinear study of interaction type. It is also checked by the obtained solution being exact and generalizable concerning Hirota's bilinear method, confirming its reliability for a new technique. This study also compares the new solutions with well-known KdV and KP equations to show that the proposed method is effective against known results and satisfactory compared to available approaches. Here, this comparison illustrates the enhancements and deviations inserted via arbitrary parameters to give a new understanding of soliton behavior. It is essential for applications where soliton behavior plays a key role, as in our case, one elementary problem can lead to N-solitons. In addition, its generality for other nonlinear equations makes it worthwhile in the large toolbox of techniques used by researchers who explore various phenomena within nonlinear dynamics.

The physical significance lies in the enhanced realism and versatility of soliton solutions with arbitrary parameters, improved understanding of soliton interactions through phase shift analysis, validation against established approaches, and the potential for broad application in studying nonlinear systems. Selecting several arbitrary parameters, we have generated N-soliton up to $N = 3$ with the given symbolic bilinear technique and analyzed the structures for the obtained solutions dynamically. We explains the analysis as follows:

- Figures 1 and 4 plot the one solitons in (a) to (c) for investigated KdV and KP equations, and analyses the soliton behavior for different values of arbitrary parameters a_0 and a_1 with the constant $\mu_1 = 0.3$ for KdV equation and $\mu_1 = 0.2, \nu_1 = -0.3, t = 0$ for KP equation. The solitons change their position with respect to the singularities depending on the parameters a_0 and a_1. Graphics (d) to (f) show the contour plots for (a) to (c), respectively. The 2D graphics (g) to (h) in figure-1 depict that the solitons are moving in right direction of x-axis.

- In figure 2 and 5, we illustrate the interactions of two solitons in (a) to (c) for investigated KdV and KP equations, and analyses the solitons behavior for different values of arbitrary parameters $a_{0,0}, a_{0,1}$ and $a_{1,0}$ with the constants $\mu_1 = 0.7, \mu_2 = 1$ for the KdV equation and $\mu_1 = \mu_2 = 0.4, \nu_1 = 0.5, \nu_2 = -0.3, t = 0$ for the KP equation. The solitons change their interaction position with respect to the singularities depending on these parameters. Graphics (d) to (f) show the contour plots for (a) to (c), respectively. The 2D graphics (g) to (h) in figure-2 depict that the solitons interactions moving in right direction of x-axis.

- Figure 3 and 6 show the interactions of three solitons in (a) to (c) for investigated KdV and KP equations, and analyze the solitons behavior for different values of arbitrary parameters $a_{0,0,0}, a_{0,0,1}, a_{0,1,0}$ and $a_{1,0,0}$ with the constants $\mu_1 = 0.7, \mu_2 = 1, \mu_3 = 0.5$ for the KdV equation and different values of constants $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3, t = 0$ for the KP equation. The soliton interactions change their position with respect to the singularities depending on these parameters. Graphics (d) to (f) show the contour plots for (a) to (c), respectively. The 2D graphics (g) to (h) in figure-3 depict that the solitons interactions moving in right direction of x-axis.

6 Conclusions

This research study analyzed the newly constructed generalized soliton solution for the well-known KdV and KP nonlinear evolution equations with a novel symbolic bilinear technique. This technique gave us an advantage in obtaining generalized soliton solutions depending on the arbitrary parameters and the constant presented in the phase variable for the investigated equations. We showed that the proposed technique establishes more generalized exact solutions than Hirota's N-solitons, which is a case with the parameter values. We investigated two well-known (1+1)-dimensional KdV and (2+1)-dimensional KP equations with the said technique and compared the obtained solutions to Hirota's soliton solutions. Generalized soliton solutions up to the third order are obtained, providing a better analysis and understanding of the solutions with arbitrary parameters. Dynamical analysis for the obtained generalized solitons has been shown through wave profiles with distinct values of the real parameters. The graphics for the first-order solution represented the single solitons. In contrast, the second and third-order solutions showed the solitons' interactions in X-type or Y-type interactions. These interactions change the positions depending on the choice of constant parameter present in phase shift. The physical significance of our research lies in the soliton solutions with more realistic and versatile arbitrary parameters. We have taken great care to ensure the validity of our results, validating them against the existing Hirota method. This validation process, along with the phase shift analysis, helps us better understand soliton interactions and provides a strong foundation for our findings. We have also discussed the phase shift and dispersion coefficient relations among arbitrary parameters, which verified the condition in Hirota's solitons solutions by choosing the values of arbitrary parameters as 1. Our analysis of the dynamic behavior of the obtained solutions with distinct parameter values, using the symbolic system *Mathematica*, further reinforces the reliability of our results.

As, the technique has been demonstrated on established equations like KdV and KP nonlinear models, there is still room for research to solve a more general class of nonlinear partial differential equations. Further research is needed to characterize how well this technique performs on computational problems that are more complex systems. These limitations are not roadblocks but point toward potential research directions. This technique offers the potential to provide generalized soliton solutions, it paves the way for significant advancements in research. The creation of generalized soliton solutions with variable parameters can enable a more adaptable and detailed explanation of physical systems, stimulating researchers and scientists intellectually. They can use this method to investigate and analyze a variety of evolution equations, leveraging the presence of arbitrary parameters to gain a deeper understanding in the fields of oceanography, plasma, fluid mechanics, water engineering, optical fibers, and other nonlinear systems.

Declarations

Ethics approval and consent to participate

Not applicable.

Competing interests

There is no conflict of interest, according to the authors.

Authors' contributions

Each author made an equal contribution to the final draft of the work. The authors would have consented and approved the final work.

Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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