A novel analysis of Cole-Hopf transformations in different dimensions, solitons and rogue waves for a (2+1)-dimensional shallow water wave equation of ion-acoustic waves in plasmas

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Abstract: This work investigates a (2+1)-dimensional shallow water wave equation of ion-acoustic waves in plasma physics. It comprehensively analyzes Cole-Hopf transformations concerning dimensions x, y, and t, and obtains the dispersion for a phase variable of this equation. We show that the soliton solutions are independent of the different logarithmic transformations for the investigated equation. We also explore the linear equations in the auxiliary function f present in Cole-Hopf transformations. We study this equation's first- and second-order rogue waves using a generalized N-rogue wave expression from the N-soliton Hirota technique. We generate the rogue waves by applying symbolic technique with β and γ as center parameters. We create rogue wave solutions for first- and second-order using direct computation for appropriate choices of several constants in the equation and center parameters. We obtain a trilinear equation by transforming variables ξ and y via logarithmic transformation for u in the function F. We harness the computational power of the symbolic tool *Mathematica* to demonstrate the graphics of the soliton and center-controlled rogue wave solutions with suitable choices of parameters. The outcomes of this study transcend the confines of plasma physics, shedding light on the interaction dynamics of ion-acoustic solitons in three-dimensional space. The equation's implications resonate across diverse scientific domains, encompassing classical shallow water theory, fluid dynamics, optical fibers, nonlinear dynamics, and many other nonlinear fields.

Keywords: Dependent-variable transformation; Logarithmic transformation; Shallow water wave; Generalized formulation; N-order rogue waves.

1 Introduction

Shallow water waves (SWWs) are an exquisite phenomenon described by waves propagating at depths significantly smaller than their wavelength [1–5]. These waves display distinguishing behaviors due to the influence of the sea or lake floor. It makes them a subject of fascination for scientists, mathematicians, and physicists. SWWs often find their origin in coastal regions with reasonably shallow water depths. Various characteristics, including wind, tides, and seismic activity, can cause them. The interaction between the wind and the water surface is a primary driver for creating interesting waves that gracefully transit the shallows.The remarkable characteristics of shallow water waves make them relevant in various applications. Coastal engineering leverages the acquaintance of SWW to design structures that can withstand wave action. Further, they play a vital role in activities such as surfing, where enthusiasts harness the energy of these waves for

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recreational purposes. Their presence not only enchants coastal landscapes but also serves as a canvas for scientific exploration and practical applications in engineering and recreation. From a scientific perspective, studying SWWs provides valuable insights into fluid dynamics and the complex interactions between water and its surroundings. Comprehending these waves is integral to forecasting coastal erosion, managing water resources, and comprehending the broader implications of climate change on coastal ecosystems.

Solitons, or solitary and self-sustaining waves, [6–9] are unprecedented phenomena in wave dynamics. Unlike conventional waves, solitons maintain their structure and energy, traveling undisturbed long distances without dissipating or losing their form. The source of solitons can be diverse, emerging from nonlinear interactions in diverse mediums such as ocean, optical fibers, and plasma. Notable for their ability to resist dispersion and sustain stability, solitons often emerge due to a delicate balance between nonlinearity and dispersion. Solitons find applications in a range of scientific and technological fields. They serve as data carriers, ensuring signal integrity over vast distances in optical communication. Their presence in fluid dynamics contributes to understanding rogue waves, and in plasma physics, they play a crucial role in sustaining plasma stability. Solitons challenge traditional wave theories, offering insights into nonlinear phenomena and the preservation of wave coherence. Their occurrence in various natural systems extends our understanding of complex wave interactions. Therefore, from revolutionizing communication technologies to unraveling mysteries in fluid dynamics, solitons stand as silent yet powerful contributors to the tapestry of scientific exploration.

Rogue waves, often called giant solitary waves emerging unexpectedly in the ocean's vastness, manifest as localized phenomena in space and time, boasting a considerable amplitude. These enigmatic occurrences, documented in sources such as [10–16], are unpredictable and ubiquitous, posing potential hazards to individuals. The exploration of the evolutionary processes behind rogue waves is imperative, capturing the attention of numerous academics. Their excessively steep height sets rogue waves apart, sometimes surpassing the magnitude of neighboring waves. This unique characteristic challenges traditional linear wave models, prompting a focus on nonlinear wave dynamics in understanding the mechanics and predicting the occurrence of these formidable waves. Rogue waves materialize randomly as tiny waves converge the energy in a confined territory. A crucial application of this research lies in enhancing marine safety. By developing models and prediction algorithms, scientists aim to provide early detection and warning systems to prevent accidents triggered by rogue waves. This knowledge significantly benefits industries such as maritime, offshore petroleum platforms, and seaside infrastructure. Consequently, a comprehensive understanding of the dynamics of rogue waves can lead to improved operational safety and cost-effective solutions, enabling the construction of safer structures and formulating strategies to mitigate their impact. Moreover, delving into the causes and dynamics of rogue waves contributes to our expanding comprehension of complicated systems, the interactions of nonlinear waves, and the emergence of severe phenomena across various fields of physics and mathematics. The study of rogue waves transcends maritime concerns, offering valuable insights into broader scientific principles and the intricacies of nonlinear wave behavior.

Researchers and scientists have studied the nonlinear partial differential equations (PDEs) or nonlinear evolution equations using several methods for obtaining the exact solutions, such as the inverse scattering method [17, 18], the Darboux transformation [19–21], the simplified Hirota's technique [22–24], Bilinear neural network method [25, 26], Hirota's bilinear method [27–30], the Lie symmetry analysis [31–33], the Bäcklund transformation [34–36], and other techniques.

This research investigates the Cole-Hopf transformations with respect to different dimensions and rogue waves for a (2+1)-dimensional SWW equation [37–39]

$$u_{tt} - u_{xx} - u_{yy} + u_x u_{xt} + u_y u_{yt} - u_{xxtt} - u_{yytt} = 0. \tag{1}$$

In 1978, Yajima N, Oikawa M. and Satsuma J. [37] modeled this equation in three dimensional interactions of ion-acoustic solitons in collisionless plasmas. They studied it using Hirota's bilinear method and discussed the one- and two-soliton solutions for the same. In the continuation, Kako F and Yajima N. [38] 1980, studied this model of ion-acoustic solitons in collisionless plasmas in two-dimensional space. They showed the dynamics for the interaction of two soliton solutions for obtained solution with some appropriate parameters. Also, they showed the interaction of two sinusoidal waves dynamically with chosen constants. In 1994, Clarkson P.A. and Mansfield E.L. [39] quoted this equation (1) in their work on a SWW equation, in which they studied a generalized SWW equation by non-Painlevé behaviour and dynamically showed the solitons' interaction solutions and breathers using Lie symmetry analysis for classical and non-classical symmetries. We found it fascinating that no more work has been done in literature on this equation as per our knowledge, whereas, this has an interesting pattern for partial derivatives in its linear terms. This gave us an idea to think for several Cole-Hopf transformations in different dimensions.

In the structure of the manuscript, next Section 2 analyzes the Cole-Hopf transformations concerning dimensions x, y, and t, and obtains the dispersion for a phase variable of investigated equation with discussion of soliton solutions for the different transformations. Section 3 studies the equation's first- and second-order solutions of rogue waves using a generalized N-rogue wave expression from the N-soliton Hirota technique with center parameters. we computes a trilinear equation in auxiliary function using the logarithmic transformation and creates rogue wave solutions up to second order for suitable values of center parameters and several constants in the equation using direct computation technique. In Section 4, we discuss the obtained solutions and the dynamical analysis. Section 5 in the end, concludes our findings and future scope.

2 Analysis of Cole-Hopf transformations

The Cole-Hopf transformation is a mathematical technique used in partial differential equations (PDEs), particularly in studying certain nonlinear PDEs. Richard Cole and Eberhard Hopf [40, 41] created the transformation in the 1950s to simplify and sometimes linearize certain types of nonlinear PDEs. It is most commonly associated with the Korteweg-de Vries (KdV) equation, a nonlinear PDE that involves nonlinear and dispersive terms and describes the propagation of long, weakly nonlinear waves, such as water waves in shallow canals. The Cole-Hopf transformation has been a valuable tool in the study of soliton theory and integrable systems, where it allows researchers to comprehend the manners of certain nonlinear wave equations and uncover essential properties, such as the existence of localized solutions, solitary waves, and several other solutions that can persist in specific nonlinear systems. The Cole-Hopf transformation, in general, is given as

$$u = R(\ln f)_{x^p} \tag{2}$$

for a given nonlinear PDE, where p represents the order of partial derivative concerning x leaning on the balance of the higher-order and nonlinear terms in the PDE.

In order to create the said transformation, we need to get the dispersion with the help of the phase variable. We consider the phase variable as

$$\alpha_i = p_i x + q_i y - w_i t, \tag{3}$$

where w_i and $p_i, q_i; i \in N$ represent dispersion and constants respectively. On substituting

$$u = e^{\alpha_i}, \tag{4}$$

into the linear terms of Eq. (1), we get the w_i as

$$w_i = \pm \frac{\sqrt{p_i^2 + q_i^2}}{\sqrt{1 - (p_i^2 + q_i^2)}}, \tag{5}$$

with $p_i^2 + q_i^2 < 1$ for getting real valued dispersion.
Now, we assume the three Cole-Hopf transformations in different dimensions x, y and t known as spatial and temporal coordinates as

$$u = u_1 = R_1(\ln f)_x, \tag{6}$$

$$u = u_2 = R_2(\ln f)_y, \tag{7}$$

$$u = u_3 = R_3(\ln f)_t, \tag{8}$$

where $R_i; i = 1, 2, 3$ are non-zero constants and $f = f(x, y, t)$ is an auxiliary function, which will be determined later. To determine the value of $R_i; i = 1, 2, 3$ in (6)-(8), we consider

$$f(x, y, t) = 1 + e^{\alpha_i} = 1 + e^{p_i x + q_i y - w_i t}. \tag{9}$$

On substituting equations (6),(7) or (8) with (9) into (1), we get the solution for R_i as

$$\begin{aligned} R_1 &= \frac{12\sqrt{p_1^2 + q_1^2}}{p_1\sqrt{1 - (p_1^2 + q_1^2)}}, \\ R_2 &= \frac{12\sqrt{p_1^2 + q_1^2}}{q_1\sqrt{1 - (p_1^2 + q_1^2)}}, \\ R_3 &= -12, \end{aligned} \tag{10}$$

for (6),(7) and (8) receptively. Thus the transformations will be as

$$\begin{aligned} u_1 &= \frac{12\sqrt{p_1^2 + q_1^2}}{p_1\sqrt{1 - (p_1^2 + q_1^2)}}(\ln f)_x, \\ u_2 &= \frac{12\sqrt{p_1^2 + q_1^2}}{q_1\sqrt{1 - (p_1^2 + q_1^2)}}(\ln f)_y, \\ u_3 &= -12(\ln f)_t. \end{aligned} \tag{11}$$

On utilizing the above transformations (11) into equation (1), we obtain transformed equations for (1) in auxiliary function f as

$$\begin{aligned} & f_{\mathrm{xtt}} f^4 - f_{\mathrm{xxx}} f^4 - f_{\mathrm{xxxtt}} f^4 - f_{\mathrm{xyy}} f^4 - f_{\mathrm{xyytt}} f^4 - f_{\mathrm{tt}} f_{\mathrm{x}} f^3 - 2 f_{\mathrm{t}} f_{\mathrm{xt}} f^3 + 3 f_{\mathrm{x}} f_{\mathrm{xx}} f^3 + 3 f_{\mathrm{xtt}} f_{\mathrm{xx}} f^3 + 6 f_{\mathrm{xt}} f_{\mathrm{xxt}} f^3 + \\ & R_1 f_{\mathrm{xx}} f_{\mathrm{xxt}} f^3 + 3 f_{\mathrm{x}} f_{\mathrm{xxtt}} f^3 + f_{\mathrm{tt}} f_{\mathrm{xxx}} f^3 + 2 f_{\mathrm{t}} f_{\mathrm{xxxt}} f^3 + R_1 f_{\mathrm{xy}} f_{\mathrm{xyt}} f^3 + f_{\mathrm{tt}} f_{\mathrm{xyy}} f^3 + 2 f_{\mathrm{t}} f_{\mathrm{xyyt}} f^3 + 2 f_{\mathrm{xy}} f_{\mathrm{y}} f^3 + 2 f_{\mathrm{xytt}} f_{\mathrm{y}} f^3 + \\ & 4 f_{\mathrm{xyt}} f_{\mathrm{yt}} f^3 + 2 f_{\mathrm{xy}} f_{\mathrm{ytt}} f^3 + f_{\mathrm{x}} f_{\mathrm{yy}} f^3 + f_{\mathrm{xtt}} f_{\mathrm{yy}} f^3 + 2 f_{\mathrm{xt}} f_{\mathrm{yyt}} f^3 + f_{\mathrm{x}} f_{\mathrm{yytt}} f^3 - 2 f_{\mathrm{x}}^3 f^2 - 12 f_{\mathrm{x}} f_{\mathrm{xt}}^2 f^2 - R_1 f_{\mathrm{t}} f_{\mathrm{xx}}^2 f^2 - R_1 f_{\mathrm{t}} f_{\mathrm{xy}}^2 f^2 - \\ & 2 f_{\mathrm{x}} f_{\mathrm{y}}^2 f^2 - 2 f_{\mathrm{xtt}} f_{\mathrm{y}}^2 f^2 - 4 f_{\mathrm{x}} f_{\mathrm{yt}}^2 f^2 + 2 f_{\mathrm{t}}^2 f_{\mathrm{x}} f^2 - 6 f_{\mathrm{x}}^2 f_{\mathrm{xtt}} f^2 - 6 f_{\mathrm{tt}} f_{\mathrm{x}} f_{\mathrm{xx}} f^2 - 12 f_{\mathrm{t}} f_{\mathrm{xt}} f_{\mathrm{xx}} f^2 - 2 R_1 f_{\mathrm{x}} f_{\mathrm{xt}} f_{\mathrm{xx}} f^2 - R_1 f_{\mathrm{x}}^2 f_{\mathrm{xxt}} f^2 - \\ & 12 f_{\mathrm{t}} f_{\mathrm{x}} f_{\mathrm{xxt}} f^2 - 2 f_{\mathrm{t}}^2 f_{\mathrm{xxx}} f^2 - 2 f_{\mathrm{t}}^2 f_{\mathrm{xyy}} f^2 - 4 f_{\mathrm{tt}} f_{\mathrm{xy}} f_{\mathrm{y}} f^2 - R_1 f_{\mathrm{xt}} f_{\mathrm{xy}} f_{\mathrm{y}} f^2 - 8 f_{\mathrm{t}} f_{\mathrm{xyt}} f_{\mathrm{y}} f^2 - R_1 f_{\mathrm{x}} f_{\mathrm{xyt}} f_{\mathrm{y}} f^2 - 8 f_{\mathrm{t}} f_{\mathrm{xy}} f_{\mathrm{yt}} f^2 - \\ & R_1 f_{\mathrm{x}} f_{\mathrm{xy}} f_{\mathrm{yt}} f^2 - 8 f_{\mathrm{xt}} f_{\mathrm{y}} f_{\mathrm{yt}} f^2 - 4 f_{\mathrm{x}} f_{\mathrm{y}} f_{\mathrm{ytt}} f^2 - 2 f_{\mathrm{tt}} f_{\mathrm{x}} f_{\mathrm{yy}} f^2 - 4 f_{\mathrm{t}} f_{\mathrm{xt}} f_{\mathrm{yy}} f^2 - 4 f_{\mathrm{t}} f_{\mathrm{x}} f_{\mathrm{yyt}} f^2 + 6 f_{\mathrm{tt}} f_{\mathrm{x}}^3 f + 6 f_{\mathrm{tt}} f_{\mathrm{x}} f_{\mathrm{y}}^2 f + \\ & 12 f_{\mathrm{t}} f_{\mathrm{xt}} f_{\mathrm{y}}^2 f + R_1 f_{\mathrm{x}} f_{\mathrm{xt}} f_{\mathrm{y}}^2 f + 2 R_1 f_{\mathrm{x}}^3 f_{\mathrm{xt}} f + 36 f_{\mathrm{t}} f_{\mathrm{x}}^2 f_{\mathrm{xt}} f + 3 R_1 f_{\mathrm{t}} f_{\mathrm{x}}^2 f_{\mathrm{xx}} f + 18 f_{\mathrm{t}}^2 f_{\mathrm{x}} f_{\mathrm{xx}} f + 12 f_{\mathrm{t}}^2 f_{\mathrm{xy}} f_{\mathrm{y}} f + 3 R_1 f_{\mathrm{t}} f_{\mathrm{x}} f_{\mathrm{xy}} f_{\mathrm{y}} f + \\ & R_1 f_{\mathrm{x}}^2 f_{\mathrm{y}} f_{\mathrm{yt}} f + 24 f_{\mathrm{t}} f_{\mathrm{x}} f_{\mathrm{y}} f_{\mathrm{yt}} f + 6 f_{\mathrm{t}}^2 f_{\mathrm{x}} f_{\mathrm{yy}} f - 2 R_1 f_{\mathrm{t}} f_{\mathrm{x}}^4 - 24 f_{\mathrm{t}}^2 f_{\mathrm{x}}^3 - 2 R_1 f_{\mathrm{t}} f_{\mathrm{x}}^2 f_{\mathrm{y}}^2 - 24 f_{\mathrm{t}}^2 f_{\mathrm{x}} f_{\mathrm{y}}^2 = 0, \end{aligned} \tag{12}$$

$$
\begin{aligned}
&-f_{\mathrm{xxytt}}f^4+f_{\mathrm{ytt}}f^4-f_{\mathrm{yyy}}f^4-f_{\mathrm{yyytt}}f^4+f_{\mathrm{tt}}f_{\mathrm{xxy}}f^3+2f_{\mathrm{t}}f_{\mathrm{xxyt}}f^3+2f_{\mathrm{x}}f_{\mathrm{xy}}f^3+2f_{\mathrm{xtt}}f_{\mathrm{xy}}f^3+4f_{\mathrm{xt}}f_{\mathrm{xyt}}f^3+R_2f_{\mathrm{xy}}f_{\mathrm{xyt}}f^3+\\
&2f_{\mathrm{x}}f_{\mathrm{xytt}}f^3-f_{\mathrm{tt}}f_{\mathrm{y}}f^3+f_{\mathrm{xx}}f_{\mathrm{y}}f^3+f_{\mathrm{xxtt}}f_{\mathrm{y}}f^3-2f_{\mathrm{t}}f_{\mathrm{yt}}f^3+2f_{\mathrm{xxt}}f_{\mathrm{yt}}f^3+f_{\mathrm{xx}}f_{\mathrm{ytt}}f^3+3f_{\mathrm{y}}f_{\mathrm{yy}}f^3+3f_{\mathrm{ytt}}f_{\mathrm{yy}}f^3+6f_{\mathrm{yt}}f_{\mathrm{yyt}}f^3+\\
&R_2f_{\mathrm{yy}}f_{\mathrm{yyt}}f^3+3f_{\mathrm{y}}f_{\mathrm{yytt}}f^3+f_{\mathrm{tt}}f_{\mathrm{yyy}}f^3+2f_{\mathrm{t}}f_{\mathrm{yyyt}}f^3-2f_{\mathrm{y}}^3f^2-R_2f_{\mathrm{t}}f_{\mathrm{xy}}^2f^2-12f_{\mathrm{y}}f_{\mathrm{yt}}^2f^2-R_2f_{\mathrm{t}}f_{\mathrm{yy}}^2f^2-2f_{\mathrm{t}}^2f_{\mathrm{xxy}}f^2-\\
&4f_{\mathrm{tt}}f_{\mathrm{x}}f_{\mathrm{xy}}f^2-8f_{\mathrm{t}}f_{\mathrm{xt}}f_{\mathrm{xy}}f^2-8f_{\mathrm{t}}f_{\mathrm{x}}f_{\mathrm{xyt}}f^2+2f_{\mathrm{t}}^2f_{\mathrm{y}}f^2-2f_{\mathrm{x}}^2f_{\mathrm{y}}f^2-4f_{\mathrm{xt}}^2f_{\mathrm{y}}f^2-4f_{\mathrm{x}}f_{\mathrm{xtt}}f_{\mathrm{y}}f^2-2f_{\mathrm{tt}}f_{\mathrm{xx}}f_{\mathrm{y}}f^2-4f_{\mathrm{t}}f_{\mathrm{xxt}}f_{\mathrm{y}}f^2-\\
&R_2f_{\mathrm{xt}}f_{\mathrm{xy}}f_{\mathrm{y}}f^2-R_2f_{\mathrm{x}}f_{\mathrm{xyt}}f_{\mathrm{y}}f^2-8f_{\mathrm{x}}f_{\mathrm{xt}}f_{\mathrm{yt}}f^2-4f_{\mathrm{t}}f_{\mathrm{xx}}f_{\mathrm{yt}}f^2-R_2f_{\mathrm{x}}f_{\mathrm{xy}}f_{\mathrm{yt}}f^2-2f_{\mathrm{x}}^2f_{\mathrm{ytt}}f^2-6f_{\mathrm{y}}^2f_{\mathrm{ytt}}f^2-6f_{\mathrm{tt}}f_{\mathrm{y}}f_{\mathrm{yy}}f^2-\\
&12f_{\mathrm{t}}f_{\mathrm{yt}}f_{\mathrm{yy}}f^2-2Rf_{\mathrm{y}}f_{\mathrm{yt}}f_{\mathrm{yy}}f^2-Rf_{\mathrm{y}}^2f_{\mathrm{yyt}}f^2-12f_{\mathrm{t}}f_{\mathrm{y}}f_{\mathrm{yyt}}f^2-2f_{\mathrm{t}}^2f_{\mathrm{yyy}}f^2+6f_{\mathrm{tt}}f_{\mathrm{y}}^3f+Rf_{\mathrm{x}}f_{\mathrm{xt}}f_{\mathrm{y}}^2f+\\
&12f_{\mathrm{t}}^2f_{\mathrm{x}}f_{\mathrm{xy}}f+6f_{\mathrm{tt}}f_{\mathrm{x}}^2f_{\mathrm{y}}f+24f_{\mathrm{t}}f_{\mathrm{x}}f_{\mathrm{xt}}f_{\mathrm{y}}f+6f_{\mathrm{t}}^2f_{\mathrm{xx}}f_{\mathrm{y}}f+3Rf_{\mathrm{t}}f_{\mathrm{x}}f_{\mathrm{xy}}f_{\mathrm{y}}f+2R_2f_{\mathrm{y}}^3f_{\mathrm{yt}}f+12f_{\mathrm{t}}f_{\mathrm{x}}^2f_{\mathrm{yt}}f+36f_{\mathrm{t}}f_{\mathrm{y}}^2f_{\mathrm{yt}}f+\\
&R_2f_{\mathrm{x}}^2f_{\mathrm{y}}f_{\mathrm{yt}}f+3R_2f_{\mathrm{t}}f_{\mathrm{y}}^2f_{\mathrm{yy}}f+18f_{\mathrm{t}}^2f_{\mathrm{y}}f_{\mathrm{yy}}f-2R_2f_{\mathrm{t}}f_{\mathrm{y}}^4-24f_{\mathrm{t}}^2f_{\mathrm{y}}^3-2R_2f_{\mathrm{t}}f_{\mathrm{x}}^2f_{\mathrm{y}}^2-f^4f_{\mathrm{xxy}}-24f_{\mathrm{t}}^2f_{\mathrm{x}}^2f_{\mathrm{y}}=0,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&f^2f_{\mathrm{tt}}-f^2f_{\mathrm{xx}}-f^2f_{\mathrm{xxtt}}-f^2f_{\mathrm{yy}}-f^2f_{\mathrm{yytt}}+4f_{\mathrm{t}}f_{\mathrm{x}}f_{\mathrm{xt}}-2f_{\mathrm{t}}^2f_{\mathrm{xx}}+2ff_{\mathrm{t}}f_{\mathrm{xxt}}+4f_{\mathrm{t}}f_{\mathrm{y}}f_{\mathrm{yt}}-2f_{\mathrm{t}}^2f_{\mathrm{yy}}+\\
&2ff_{\mathrm{t}}f_{\mathrm{yyt}}-ff_{\mathrm{t}}^2-2f_{\mathrm{tt}}f_{\mathrm{x}}^2+ff_{\mathrm{tt}}f_{\mathrm{xx}}-2f_{\mathrm{tt}}f_{\mathrm{y}}^2+ff_{\mathrm{tt}}f_{\mathrm{yy}}+2ff_{\mathrm{x}}f_{\mathrm{xtt}}+ff_{\mathrm{x}}^2-4ff_{\mathrm{xt}}^2+2ff_{\mathrm{y}}f_{\mathrm{ytt}}+ff_{\mathrm{y}}^2-4ff_{\mathrm{yt}}^2=0,
\end{aligned}
\tag{14}
$$

for (6),(7) and (8) receptively.
Considering the function f in any of the above equation as

$$f=1+e^{\alpha_1}=1+e^{p_1x+q_1y-\frac{\sqrt{p_i^2+q_i^2}}{\sqrt{1-(p_i^2+q_i^2)}}t}. \tag{15}$$

Thus, by putting up the expression for f from (15) into any transformation of u in (11), we get the same solution as

$$u=\frac{12\sqrt{p_1^2+q_1^2}e^{p_1x+q_1y}}{\sqrt{1-(p_1^2+q_1^2)}\left(e^{\frac{t\sqrt{p_1^2-q_1^2}}{\sqrt{1-(p_1^2+q_1^2)}}}+e^{p_1x+q_1y}\right)}, \tag{16}$$

which shows that the (2+1)-dimensional SWW equation (1) is independent of the Cole-Hopf transformations and any transformation can be used to get the soliton solutions. Readers can follow the works [37, 38] to dive into the soliton solutions and their interactions.

3 Center-controlled rogue waves

On transforming $u=u(\xi,y)$ with $\xi=x-h*t$ the equation (1), we get

$$h^2(u_{\xi\xi}-u_{\xi\xi\xi\xi}-u_{\xi\xi\mathrm{yy}})-h(u_\xi u_{\xi\xi}+u_{\xi\mathrm{y}}u_{\mathrm{y}})-u_{\xi\xi}-u_{\mathrm{yy}}=0. \tag{17}$$

Considering the phase Φ_i in Eq. (17) as

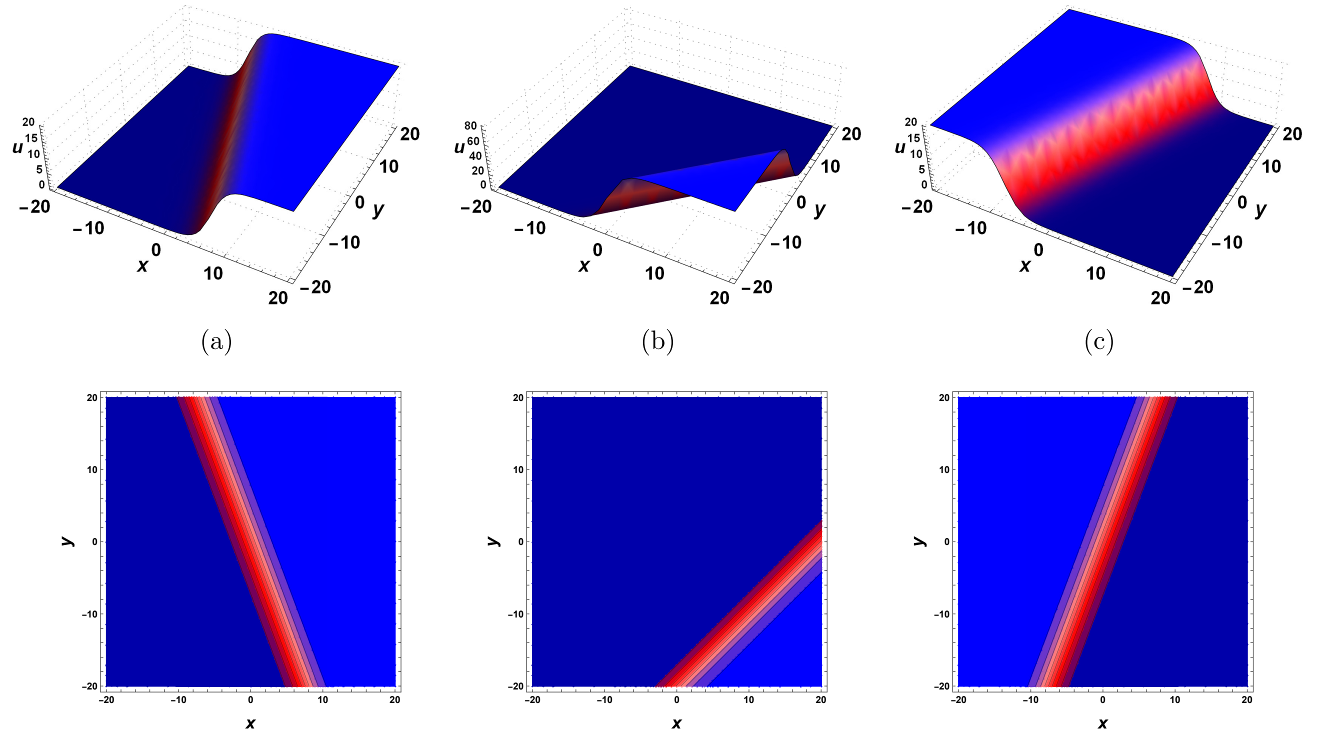
$$\Phi_i=p_i\xi-w_iy, \tag{18}$$

where p_i and $w_i; i\in N$ are parameters and dispersion, respectively. By putting $u(\xi,y)=e^{\Phi_i}$ in Eq. (17) for linear terms, we obtain

$$w_i=\pm\frac{\sqrt{-h^2p_i^4+h^2p_i^2-p_i^2}}{\sqrt{h^2p_i^2+1}}. \tag{19}$$

We take the dependent variable transformation as

$$u(\xi,y)=R(\log F)_\xi, \tag{20}$$



(a) (b) (c)

(d) (e) (f)

Figure 1: Solitons for the solution (16) with (15) having values **(a)** $p_1 = 0.8, q_1 = 0.3$; **(b)** $p_1 = 0.7, q_1 = -0.7$; and **(c)** $p_1 = -0.8, q_1 = 0.3$. **(d)-(f)** are 2D outlines for **(a)-(c)** concerning contours in ξy-plane.

and put it in Eq. (17) with Eq. (19) and $f = 1 + e^{\Phi_1}$, then we get R as

$$R = 12h.$$

So, the equation (20) becomes

$$u(\xi, y) = u_0 + 12h(lnF)_\xi, \tag{21}$$

where u_0 is a constant parameter. On substituting Eq. (21) into (17), we get a trilinear equation in $F(\xi, y)$ as

$$F^2h^2F_{\xi\xi\xi\xi} - F^2h^2F_{\xi\xi} + F^2h^2F_{\xi\xi\mathrm{yy}} + F^2F_{\xi\xi} + F^2F_{\mathrm{yy}} - 4Fh^2F_\xi F_{\xi\xi\xi} - 2Fh^2F_\xi F_{\xi\mathrm{yy}} + Fh^2F_\xi^2 + \\ 3Fh^2F_{\xi\xi}^2 + 4Fh^2F_{\xi\mathrm{y}}^2 - 4h^2F_\xi F_{\xi\mathrm{y}}F_\mathrm{y} + 2h^2F_{\xi\xi}F_\mathrm{y}^2 - 2Fh^2F_{\xi\xi\mathrm{y}}F_\mathrm{y} + 2h^2F_\xi^2F_{\mathrm{yy}} - Fh^2F_{\xi\xi}F_{\mathrm{yy}} - FF_\xi^2 - FF_\mathrm{y}^2 = 0. \tag{22}$$

In 2023, Kumar S. and Mohan B. [42] generalized a direct technique to construct N-order rogue waves using N-soliton solution in Hirota's technique which was firstly discussed by X. Yang, et al. [43] in 2022, for a (3+1)-dimensional KdV-BBM equation with limit technique of long wave. It gives the generalized representation of N-rogue waves as

$$F_N(\xi, y) = \sum_{j=0}^{\frac{N^2+N}{2}} \sum_{k=0}^{j} s_{N^2+N-2j,2k}(y)^{2k}(\xi)^{N^2+N-2j}, \tag{23}$$

which has a resemblance to the functions used in symbolic computational approach [44,45] by Zhaqilao [46]. This can be expressed with center controlled parameter as

$$F(\xi, y) = \widehat{F_N}(\xi, y, \beta, \gamma) = \sum_{j=0}^{\frac{N^2+N}{2}} \sum_{k=0}^{j} s_{N^2+N-2j,2k}(y-\gamma)^{2k}(\xi-\beta)^{N^2+N-2j}, \tag{24}$$

where $s_{i,j}; i,j \in \{0, 2, \cdots, j(j+1)\}$ are constants and (β, γ) are center parameters.

3.1 First-order solution of rogue waves

Considering $F(\xi, y)$ with $N = 1$ in Eq. (24) as

$$F(\xi, y) = s_{2,0}\xi^2 + s_{0,2}y^2 + s_{0,0}, \tag{25}$$

and substituting it into (22) gives a system by equating the coefficients for distinct powers of $\xi^m y^n; m, n \in Z$ to zero as

$$\begin{aligned} 2h^2 s_{2,0}^3 - 2s_{2,0}^3 + 2s_{0,2}s_{2,0}^2 &= 0, \\ 12h^2 s_{2,0}^3 + 12h^2 s_{0,2}s_{2,0}^2 + 4s_{0,0}s_{0,2}s_{2,0} &= 0, \\ 12h^2 s_{0,2}s_{2,0}^2 + 12h^2 s_{0,2}^2 s_{2,0} - 4h^2 s_{0,0}s_{0,2}s_{2,0} + 4s_{0,0}s_{0,2}s_{2,0} &= 0. \end{aligned} \tag{26}$$

Solving above system gives constants as

$$s_{0,0} = \frac{3h^2\left(h^2-2\right)s_{2,0}}{1-h^2}, \quad s_{0,2} = \left(1-h^2\right)s_{2,0}, \quad s_{2,0} = s_{2,0}. \tag{27}$$

Thus, the equation (25) with (27) will be as

$$F(\xi, y) = \widehat{f}_1(\xi, y, \beta, \gamma) = s_{2,0}\left((\beta-\xi)^2 + \left(1-h^2\right)(y-\gamma)^2 + \frac{3\left(h^2-2\right)h^2}{1-h^2}\right), \tag{28}$$

which gives a solution of Eq. (22). We get a first-order solution of rogue waves on substituting Eq. (28) into (21) as

$$u(\xi, y) = u_0 + \frac{24h(\xi-\beta)}{(\beta-\xi)^2 + (1-h^2)(y-\gamma)^2 + \frac{3(h^2-2)h^2}{1-h^2}}, \tag{29}$$

3.2 Second-order solution of rogue waves

Taking auxiliary function $F(\xi, y)$ with $N = 2$ in Eq. (24) as

$$F(\xi, y) = s_{6,0}\xi^6 + s_{4,0}\xi^4 + s_{2,0}\xi^2 + s_{0,6}y^6 + s_{2,4}\xi^2 y^4 + s_{0,4}y^4 + s_{4,2}\xi^4 y^2 + s_{2,2}\xi^2 y^2 + s_{0,2}y^2 + s_{0,0}, \tag{30}$$

and put it in trilinear Eq. (22). On equating zero the coefficients for distinct powers of $\xi^m y^n; m, n \in Z$, we obtain a system, which gives the constant values as

$$s_{0,0} = \frac{h^6\left(286h^6 - 1383h^4 + 2548h^2 - 2076\right)s_{4,2}}{\left(h^2-1\right)^4}, \quad s_{0,2} = \frac{h^4\left(144h^4 - 627h^2 + 958\right)s_{4,2}}{3\left(h^2-1\right)^2},$$

$$s_{0,4} = \frac{1}{3}h^2\left(33h^2 - 50\right)s_{4,2}, \quad s_{0,6} = \frac{1}{3}\left(h^2-1\right)^2 s_{4,2}, \quad s_{2,0} = -\frac{h^4\left(144h^4 - 507h^2 + 238\right)s_{4,2}}{3\left(h^2-1\right)^3},$$

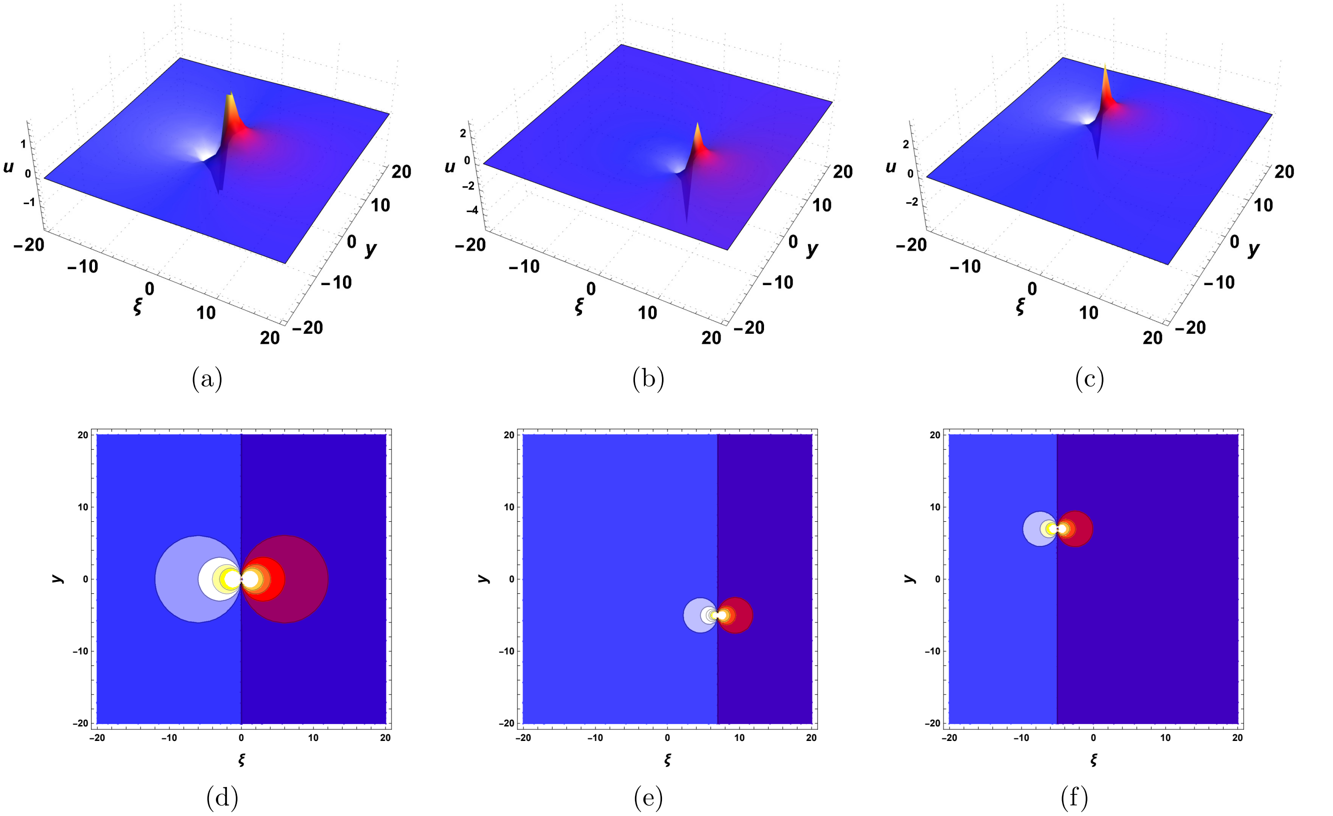


Figure 2: Rogue waves for first-order solution (29) with (28) having values **(a)** $u_0 = 0, h = 0.1, \beta = \gamma = 0$; **(b)** $u_0 = 0, h = 0.1, \beta = 7, \gamma = -6$; and **(c)** $u_0 = 0, h = 0.1, \beta = -5, \gamma = 7$. **(d)-(f)** are 2D outlines for **(a)-(c)** concerning contours in ξy-plane.

$$s_{2,2} = -\frac{6h^2\left(h^2-6\right)s_{4,2}}{h^2-1}, \quad s_{2,4} = -\left(h^2-1\right)s_{4,2}, \quad s_{4,0} = \frac{h^2\left(33h^2-58\right)s_{4,2}}{3\left(h^2-1\right)^2}, \quad s_{4,2} = s_{4,2},$$
$$s_{6,0} = \frac{s_{4,2}}{3(1-h^2)}. \tag{31}$$

So, the Eq. (25) with (31) becomes

$$F(\xi,y) = \widehat{f}_2(\xi,y,\beta,\gamma) = \frac{s_{4,2}}{3}\Big(\frac{\left(33h^2-58\right)h^2(\beta-\xi)^4}{\left(h^2-1\right)^2} - \frac{(\beta-\xi)^6}{h^2-1} - \frac{18\left(h^2-6\right)h^2(\beta-\xi)^2(y-\gamma)^2}{h^2-1} -$$
$$3\left(h^2-1\right)(\beta-\xi)^2(y-\gamma)^4 + \left(33h^2-50\right)h^2(y-\gamma)^4 + \left(h^2-1\right)^2(y-\gamma)^6 - \frac{\left(144h^4-507h^2+238\right)h^4(\beta-\xi)^2}{\left(h^2-1\right)^3} +$$
$$\frac{\left(144h^4-627h^2+958\right)h^4(y-\gamma)^2}{\left(h^2-1\right)^2} + \frac{3\left(286h^6-1383h^4+2548h^2-2076\right)h^6}{\left(h^2-1\right)^4} + 3(\beta-\xi)^4(y-\gamma)^2\Big), \tag{32}$$

which gives a solution of Eq. (22). We get a second-order solution of rogue waves on substituting Eq. (32) into (21) as

$$u(\xi,y) = u_0 + 12h(\ln \widehat{f}_2(\xi,y,\beta,\gamma))_\xi. \tag{33}$$

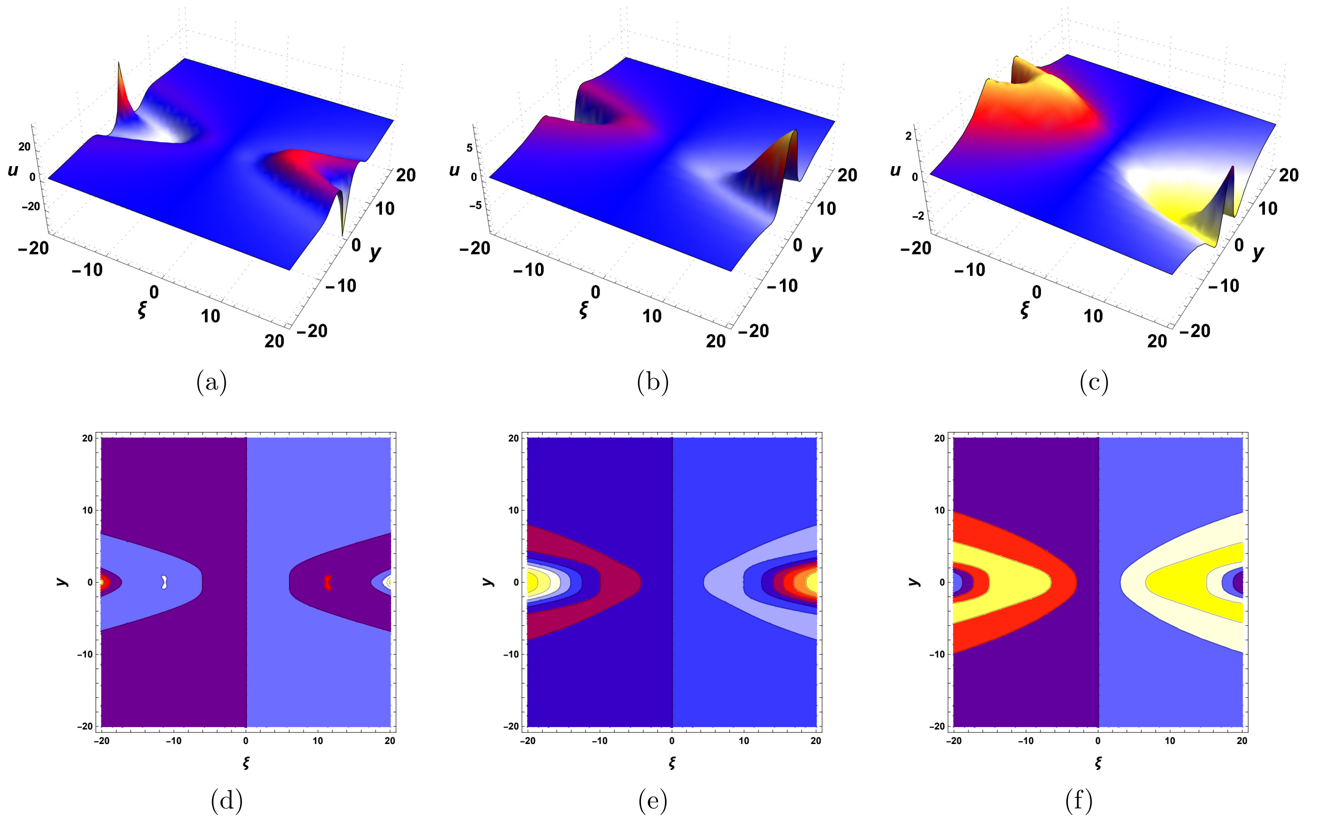


Figure 3: Rogue waves for second-order solution (33) with (32) having values **(a)** $u_0 = 0, h = 4, \beta = \gamma = 0$; **(b)** $u_0 = 0, h = 8, \beta = \gamma = 0$; and **(c)** $u_0 = 0, h = 12, \beta = \gamma = 0$. **(d)-(f)** are 2D outlines for **(a)-(c)** concerning contours in ξy-plane.

4 Results and discussion

Our investigation shows that the (2+1)-dimensional SWW equation governing ion-acoustic waves in plasma physics can have different Cole-Hopf transformations in different dimensions x, y, and t. The analysis of these transformations showed that the soliton solutions for this SWW equation are independent of the Cole-Hopf transformations and give the same solution as discussed in Section 2. By selecting the appropriate parameters, we found the first- and second-order solutions of rogue waves with center parameters (β, γ) with the said symbolic approach and dynamically showed the graphics of the obtained solutions. Therefore, we explain the results and findings as follows:

- In Figure 1, we illustrate the solitons for the solution (16) with (15) with respect to the singularity about the x axis. (a)-(c) show the dynamics of solitons with values **(a)** $p_1 = 0.8, q_1 = 0.3$; **(b)** $p_1 = 0.7, q_1 = -0.7$; and **(c)** $p_1 = -0.8, q_1 = 0.3$.

- Figure 2 depicts the first-order solution of rogue waves with center parameters (β, γ). It shows single rogue waves concerning singularity through center parameters (β, γ) with values **(a)** $u_0 = 0, h = 0.1, \beta = \gamma = 0$; **(b)** $u_0 = 0, h = 0.1, \beta = 7, \gamma = -5$; and **(c)** $u_0 = 0, h = 0.1, \beta = -5, \gamma = 7$.

- In Figure 3, we show the second-order solution of rogue waves with center parameters (β, γ). Dynamics shows that the direction and amplitude of rogue waves depend on the transforming parameter h in

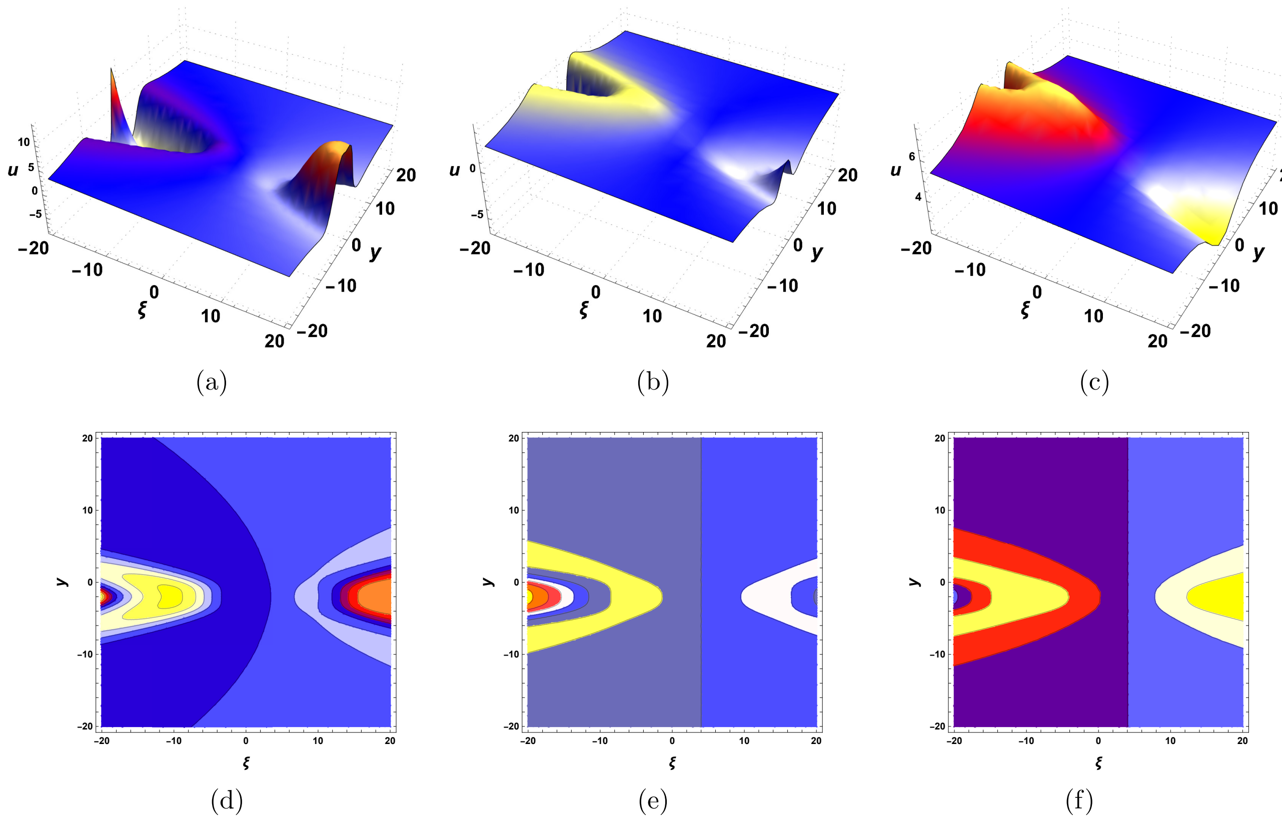


Figure 4: Rogue waves for second-order solution (33) with (32) having values **(a)** $u_0 = 2, h = 5, \beta = 4, \gamma = -2$; **(b)** $u_0 = 2, h = 10, \beta = 4, \gamma = -2$; and **(c)** $u_0 = 5, h = 15, \beta = 4, \gamma = -2$. **(d)-(f)** are 2D outlines for **(a)-(c)** concerning contours in ξy-plane.

$\xi = x - ht$. Rogue waves are plotted with values **(a)** $u_0 = 0, h = 4, \beta = \gamma = 0$; **(b)** $u_0 = 0, h = 8, \beta = \gamma = 0$; and **(c)** $u_0 = 0, h = 12, \beta = \gamma = 0$.

- Figure 4 shows the second-order solution of rogue waves with center parameters (β, γ). It shows that the direction and amplitude of rogue waves depend on the transforming parameter h in $\xi = x - ht$. Rogue waves are plotted with values **(a)** $u_0 = 2, h = 5, \beta = 4, \gamma = -2$; **(b)** $u_0 = 2, h = 10, \beta = 4, \gamma = -2$; and **(c)** $u_0 = 5, h = 15, \beta = 4, \gamma = -2$.

5 Conclusions

In conclusion, our investigation of the (2+1)-dimensional SWW equation governing ion-acoustic waves in plasma physics has revealed analytical insights and dynamic phenomena. Through a meticulous analysis of Cole-Hopf transformations in dimensions x, y, and t, we have derived the dispersion relation for the phase variable and illustrated soliton solutions that remain unaffected by these transformations. Our investigation extends to rogue waves, delving into first- and second-order occurrences using a generalized N-rogue wave expression derived by the N-soliton in Hirota technique. Application of symbolic computation, notably the center parameters β and γ, has allowed us to formulate rogue wave solutions, offering a subtle understanding between the parameters and the resulting dynamics. By employing direct computation for various parameter values and reasonable choices of constants, we have manifested solutions of rogue waves up to second-order with their dynamics. Moreover, our exploration incorporates a logarithmic transformation for the depen-

dent variable u, leading to a trilinear equation in $F(\xi, y)$. In practical terms, our findings resonate across diverse scientific disciplines, ranging from classical shallow water theory and fluid dynamics to optical fibers and nonlinear dynamics. The three-dimensional space investigated in the context of ion-acoustic solitons in plasmas holds promise for real-world applications, offering insights that transcend the boundaries of plasma physics.
This research contributes to the theoretical understanding of (2+1)-dimensional SWW equation with practical applications in diverse nonlinear fields. The dynamics, soliton solutions, and rogue wave occurrences uncovered in this study provide a solid foundation for future investigations and underline the rich potential of this area of research.

Conflict of interest

The authors have no conflicts to disclose.

Data availability statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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