Painlevé analysis, restricted bright-dark N-solitons, and N-rogue waves of a (4+1)-dimensional variable-coefficient generalized KP equation in nonlinear sciences

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Abstract: This research work studies the complete integrability, bright-dark solitons, and rogue waves of a recently formed variable coefficient generalized (4+1)-dimensional Kadomtsev-Petviashvili equation. It analyses the integrability of the investigated generalized equation by applying the Painlevé test with arbitrary choices and fulfilling the condition for compatibility for the resonances. It generates the bilinear equation with the Cole-Hopf transformation in the auxiliary function and, by using the bilinear differential operator, construct Hirota's bilinear form of this equation. Utilizing Hirota's bilinear technique for N-soliton solutions, we obtain soliton solutions and their X-type and Y-type interactions for 1-, 2-, and 3-soliton solutions under the obtained restrictions and showcase their analytic dynamics. Also, it obtains the N-rogue wave solutions up to second order with center-controlled parameters with appropriate parameters and the variable coefficients and display the dynamical structures. We form the bright-dark solitons and rogue waves with appropriate choices of parameters in the third and second order, respectively. By applying the computer algebra system software *Mathematica*, it displays the dynamical structures for the generated solutions with several chosen parameter values. Solitons appear in different fields of nonlinear sciences, such as fluid mechanics, nonlinear optics, oceanography, plasma physics, water waves, and other sciences.

Keywords: Hirota bilinear method; Integrability; Cole-Hopf transformation; Symbolic computational approach; Dispersion relation.

1 Introduction

Recently, in 2021, Fan and Bao proposed a (4+1)-dimensional equation in fluid mechanics named as variable-coefficient generalized Kadomtsev–Petviashvili (vc-gKP) equation [1]:

$$u_{xt}+A(t)\left(uu_x\right)_x+B(t)u_{xxxx}+C_1(t)u_{xx}+C_2(t)u_{xy}+C_3(t)u_{xz}+C_4(t)u_{xs}+C_5(t)u_{yy}+C_6(t)u_{zz}+C_7(t)u_{ss}=0, \tag{1}$$

where $u(x,y,z,s,t)$ is a wave amplitude of x,y,z,s and t coordinates. The functions $A(t)$ and $B(t)$ are the coefficients of the nonlinear and dispersive terms, respectively. $C_i(t) : 1 \leq i \leq 4$ are the time-dependent coefficient functions for the perturbed terms; and $C_i(t) : 5 \leq i \leq 7$ are the coefficient functions for the disturbed terms of wave velocities in y,z and s directions. Using variable coefficients in the (4+1)-dimensional variable coefficient generalized Kadomtsev-Petviashvili (vc-gKP) equation offers significant advantages in modeling complex real-world physical systems. Many systems, such as fluids, plasmas, and optical fibers, exhibit non-uniform properties like nonlinearity and dispersion that vary over space and time. The equation

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can accurately capture these dynamic environments by incorporating variable coefficients, offering a more realistic representation of wave behavior. This adaptability allows the equation to simulate intricate phenomena such as solitons, rogue waves, and breathers. Moreover, adjusting coefficients provides flexibility in tuning wave properties like amplitude and speed, enabling researchers to explore a broader range of behaviors. It also can uncover localized solutions, such as rogue waves, which can intrigue researchers. This feature helps better understand soliton interactions and stability, which are critical in various applications like oceanography, plasma physics, and optical communication.
Fan and Bao studied the lump, e-lump solutions and their interaction as lump-soliton, rogue-soliton, lump-kink and e-lump-soliton solutions. Although, their work shows the novel solutions and conclude their findings, still the equation (1) has potential opportunity to add some focus on the key points as

- The integrability of the proposed vc-gKP equation
- Illustration of dispersion using phase variable
- Construction of Cole-Hopf transformation and bilinear equation
- N-soliton solutions via Hirota bilinear approach
- N-rogue waves with center-parameters via symbolic computational approach

This research work focuses on the above said points to investigate the complete integrability with Painlevé analysis [2, 3]; evaluation of dispersion as a function of temporal coordinate by considering the appropriate phase variable; formulation of Cole-Hopf transformation or logarithmic transformation using obtained dispersion and then get the bilinear equation and its Hirota's D-operator bilinear form; construction of N-solitons by applying the Hirota's bilinear method for $N = 1, 2, 3$.
The equation (1) generalizes different well-known equations

- For $u = u(x, y, z, t), C_4(t) = C_7(t) = 0$, equation (1) becomes a (3+1)-dimensional vc-KP equation [4] as

$$u_{xt} + A(t)\left(uu_x\right)_x + B(t)u_{xxxx} + C_1(t)u_{xx} + C_2(t)u_{xy} + C_3(t)u_{xz} + C_5(t)u_{yy} + C_6(t)u_{zz} = 0. \tag{2}$$

- For $u = u(x, y, t), C_3(t) = C_4(t) = 0, C_6(t) = C_7(t) = 0$, equation (1) gives a (2+1)-dimensional vc-KP equation [5] as

$$u_{xt} + A(t)\left(uu_x\right)_x + B(t)u_{xxxx} + C_1(t)u_{xx} + C_2(t)u_{xy} + C_5(t)u_{yy} = 0. \tag{3}$$

- For $u = u(x, y, t), C_i(t) = 0; 1 \leq i \leq 4, C_j(t) = 0; 6 \leq j \leq 7$, equation (1) reduces to a (2+1)-dimensional vc-KP equation [6] as

$$u_{xt} + A(t)\left(uu_x\right)_x + B(t)u_{xxxx} + C_5(t)u_{yy} = 0. \tag{4}$$

- For $u = u(x, t), A(t) = 6, B(t) = 1, C_i(t) = 0; 1 \leq i \leq 7$, equation (1) becomes the KdV equation [7] as

$$u_t + 6uu_x + u_{xxx} = 0. \tag{5}$$

Soliton [8–15] as a water wave phenomenon presented in various physics areas occurs by having a balance between nonlinearity and dispersion. They preserve their shape and maintain stability over extensive distances. Solitons, or solitary waves [16–22], are essential elements in numerous dynamical systems, such as

fiber-optic communication, balancing optical-fiber dispersion, and nonlinearity. The science, technology, engineering, and mathematics (STEM) fields have a practical presence, such as oceanography, the engineering of coastlines, and wave energy conversion. Understanding the formation and behavior of the solitons can provide steadfast support to address coastal erosion, optimize wave energy extraction, and improve wave forecasting. Solitons in plasma physics are advantageous for studying the propagation of waves in magnetized environments, fusion studies, and several research areas.
Rogue waves [23–28] are unexpected and come suddenly into the ocean. They are localized with a large amplitude in space-time. Nonlinear sciences study these types of waves for their contradicting traditional linear models. Research in nonlinear waves studies the occurrence of rogue waves to reduce the harm to individuals with forecasting. Marine safety is one of the keys to studying such large amplitude waves, which can be reduced by providing early warning and detection to prevent mishappening by developing prediction algorithms and techniques. Therefore, studying rogue waves plays a vital role in understanding complex models of nonlinear waves in diverse mathematical and nonlinear sciences.
Nonlinear PDE models the complex physical systems in various scientific physics and applied mathematics domains. Scientists have used PDEs to solve conjectures such as Poincare's and Calabi's by demonstrating a variety of physical phenomena, from nonlinear dynamics to gravity. It is challenging to study nonlinear PDEs because there are no general techniques for their investigation. Usually, each equation needs to be examined independently as a problem. However, there are some circumstances in which broad approaches are appropriate. Several techniques are being used to obtain the analytical and exact solutions: the Darboux transformation [29,30]; simplified Hirota's technique [31,32]; the Bäcklund transformation [33,34]; the bilinear neural network method [35,36]; Hirota's bilinear approach [37–39]; symmetry analysis [40,41]; and others.

The structure of the manuscript follows as: The following section analyses the Painlevé integrability of the investigated vc-gKP equation by applying the Painlevé test. Section 3 constructs the Cole-Hopf transformation by finding the dispersion by considering a phase variable. Further, it shows the bilinearity in the auxiliary function and Hirota's D-operator form for the studied equation. In Section 4, we obtain the N-solitons using Hirota's bilinearization technique up to third-order soliton solutions and their interactions under the restrictions. It also displays the dynamical structures while considering appropriate parameter values. Section 5 obtains the N-rogue waves with center parameters up to second order and displays the obtained solution's dynamic structures with appropriate parameter values. Section 6 analyzes the obtained solutions, and the last section concludes this study.

2 Integrability test

The Painlevé test [42, 43] is a powerful technique to examine the integrability of a nonlinear PDE. This analysis aims to determine whether an investigated PDE forms solutions without movable singularities. A singularity is movable if it can be removed or changed by applying the proper coordinate transformation. When a PDE clears the integrability test of Painlevé, it is said to be P-integrable or Painlevé integrable, indicating that it has a complex structure and provides solutions with the help of specialized functions. Painlevé analysis requires the Laurent series expansion around a singularity that is movable for the PDE solutions. We insert this series into the PDE and equate the coefficients of several powers to obtain a set of consistency conditions. If the test fails, it implies that there may be non-generic singularities in the PDE, or its Painlevé integrability may be lacking. Since it highlights distinct solutions and uncovers previously hidden integrable structures, it is a successful technique for analyzing nonlinear PDEs. By identifying integrable PDEs, researchers can develop analytical strategies and techniques to solve these PDEs and gain a deeper understanding of the complex physical phenomena of nonlinear models.
On verifying integrable conditions, Weiss-Tabor-Carnevale (WTC) [43] provided the Painlevé analysis to

assess the integrability of the nonlinear PDEs. This analysis has three steps: examining leading-order behavior, finding resonances, and assuring resonance conditions. With $g = 0$, the singular manifold , we consider the Laurent series for the function u as

$$u(x, y, z, s, t) = \sum_{\xi=0}^{\infty} u_{\xi} g^{\xi+\Lambda}, \tag{6}$$

where Λ and u_{ξ} are the negative integer and the arbitrary functions, respectively. Substituting the equation (6) into the equation (1) gives the values as

$$\Lambda = -2,$$

and

$$u_0 = -\frac{12B(t)g_{\mathrm{x}}^2}{A(t)}.$$

Using the above values, we find the equation for resonances

$$(\xi - 6)(\xi - 5)(\xi - 4)(\xi + 1)A(t)B(t)g_x^4 = 0, \tag{7}$$

that concludes the resonances

$$\xi = -1, \quad 4, \quad 5, \quad 6.$$

Here, $\xi = -1$ occurs for the arbitrary manifold $g = 0$. We obtain u_{ξ} as explicit functions for $0, 1, 2,$ and 3 and arbitrary functions for $\xi = 4, 5,$ and 6 as

$$\begin{aligned} u_1 &= \frac{12B(t)g_{xx}}{A(t)}, \\ u_2 &= -\frac{B(t)(4g_x g_{xxx} - 3g_{xx}^2) + g_t g_x + P}{A(t)g_x^2}, \\ u_3 &= \frac{1}{A(t)g_x^4}(-g_x g_{xx} Q + g_{xx} R + g_x^2 S), \end{aligned}$$

$u_4 = u_4, \qquad u_5 = u_5, \qquad u_6 = u_6,$
where

$$P = C_1(t)g_x^2 + C_2(t)g_x g_y + C_3(t)g_x g_z + C_4(t)g_s g_x + C_5(t)g_y^2 + C_6(t)g_z^2 + C_7(t)g_s^2,$$

$$Q = 4B(t)g_{xxx} + C_2(t)g_y + C_3(t)g_z + C_4(t)g_s + g_t,$$

$$R = 3B(t)g_{xx}^2 - C_5(t)g_y^2 - C_6(t)g_z^2 - C_7(t)g_s^2,$$

$$S = B(t)g_{xxxx} + C_2(t)g_{xy} + C_3(t)g_{xz} + C_4(t)g_{xs} + C_5(t)g_{yy} + C_6(t)g_{zz} + C_7(t)g_{ss} + g_{xt}.$$

We observe that the resonances $\xi = 4, 5,$ and 6 identically satisfy the compatibility conditions. Thus, the investigated equation (1) passes the test for complete integrability.

3 Cole-Hopf transformation and bilinear form

The Cole-Hopf transformation is a mathematical method for studying PDEs, especially nonlinear PDEs. This transformation was developed in the 1950s by Richard Cole and Eberhard Hopf [44,45] to simplify and occasionally linearize specific kinds of nonlinear PDEs. It is most frequently linked to the KdV equation, a nonlinear PDE that describes the propagation of long, weakly nonlinear waves, like water waves in shallow canals, and contains nonlinear and dispersive terms. In the study of soliton theory and integrable systems, the Cole-Hopf transformation has proven invaluable. It enables researchers to understand the behavior of some nonlinear wave equations and reveal crucial properties, including the existence of localized solutions, solitary waves, and considerably other solutions that may endure in particular nonlinear systems. In general, the Cole-Hopf transformation is defined as

$$u = P(\ln f)_{px}, \tag{8}$$

where u is the dependent variable, P is a constant and p represents the order of partial derivative concerning x that leans on the balancing between nonlinear and higher-order terms in the PDE.
We take κ_i as the phase in the equation (1) as

$$\kappa_i = g_i x + h_i y + p_i z - q_i s - w_i(t), \tag{9}$$

with $g_i, h_i, p_i, \text{and} q_i$ for $i = 1, 2, ...,$ as constant parameters. $w_i(t)$ represent the dispersion coefficients. On putting $u = \mathrm{e}^{\kappa_i}$ in linear terms of equation (1), we get

$$w_i = \int \left(\frac{B(t)g_i^4 + C_1(t)g_i^2 + C_2(t)g_i h_i + C_3(t)g_i p_i + C_4(t)g_i q_i + C_5(t)h_i^2 + C_6(t)p_i^2 + C_7(t)q_i^2}{g_i} \right) dt. \tag{10}$$

Considering the transformation

$$u = P(\ln f)_{xx}, \tag{11}$$

and inserting it with $f(x, y, z, s, t) = 1 + \mathrm{e}^{\kappa_1}$ and dispersion relation (10) into equation (1), we get P as

$$P = \frac{12B(t)}{A(t)}.$$

The transformation (11) transforms the equation (1) into a bilinear equation in f as

$$\begin{aligned} &kA(t)(4f_x f_{xxx} - 3f_{xx}^2 - f f_{xxxx}) + C_1(t)(f_x^2 - f f_{xx}) + C_2(t)(f_x f_y - f f_{xy}) + C_3(t)(f_x f_z - f f_{xz}) \\ &+ C_4(t)(f_s f_x - f f_{xs}) + C_5(t)(f_y^2 - f f_{yy}) + C_6(t)(f_z^2 - f f_{zz}) + C_7(t)(f_s^2 - f f_{ss}) + (f_t f_x - f f_{xt}) = 0, \end{aligned} \tag{12}$$

under the constraint

$$\frac{B(t)}{A(t)} = k \quad k \neq 0. \tag{13}$$

Thus the dependent variable transformation (11) with constraint (13) is

$$u = 12k(\ln f)_{xx} \quad k \neq 0. \tag{14}$$

To construct the bilinear D-operator form of the equation (1), we use the bilinear differential operators $D_i : i = x, y, z, s, t$ defined by Hirota [7] as

$$D_x^{R_1} D_y^{R_2} D_z^{R_3} D_s^{R_4} D_t^{R_5} U(x, y, z, s, t)V(x, y, z, s, t) = \left(\frac{\partial}{\partial_x} - \frac{\partial}{\partial_{x'}} \right)^{R_1} \left(\frac{\partial}{\partial_y} - \frac{\partial}{\partial_{y'}} \right)^{R_2} \left(\frac{\partial}{\partial_z} - \frac{\partial}{\partial_{z'}} \right)^{R_3}$$

$$\times \left(\frac{\partial}{\partial_s} - \frac{\partial}{\partial_{s'}} \right)^{R_4} \left(\frac{\partial}{\partial_t} - \frac{\partial}{\partial_{t'}} \right)^{R_5} U(x, y, z, s, t)V(x', y', z', s', t')|_{x=x', y=y', z=z', s=s', t=t'},$$

with x', y', z', s', and t' as the formal variables and $R_j : 1 \leq j \leq 5$ as positive integers. With this D-operator definition, we get the required operators as

$$\begin{aligned} D_x D_w f.f &= 2(f f_{xw} - f_x f_w); \quad w : y, z, s, t \\ D_w^2 f.f &= 2(f f_{ww} - f_w^2); \quad w : x, y, z, s, t \\ D_x^4 f.f &= 2(3f_{xx}^2 - 4f_x f_{xxx} + f f_{xxxx}). \end{aligned} \tag{15}$$

Thus, the bilinear equation (12) can be written in the Hirota's bilinear form with bilinear differentials D as

$$[kA(t)D_x^4+C_1(t)D_x^2+C_2(t)D_xD_y+C_3(t)D_xD_z+C_4(t)D_xD_s+C_5(t)D_y^2+C_6(t)D_z^2+C_7(t)D_s^2+D_xD_t]f.f=0. \tag{16}$$

4 N-soliton solutions

For N-soliton solution, consider the expression for f given by Hirota, in bilinear equation (16) as

$$f=\sum_{\mu=0,1}\exp\left(\sum_{i=1}^{N}\mu_i\kappa_i+\sum_{i=1<j}^{N}A_{ij}\mu_i\mu_j\right), \tag{17}$$

where $\sum_{\mu=0,1}$ indicates the summation of all possible combinations for $\mu_i=0,1$ for $1\le i\le N$.
For $N=1$, we have $\mu_1=0,1$ so $f=1+\mathrm{e}^{\kappa_1}$.
For $N=2$, we have $\mu_1,\mu_2=0,1$. So there will be four combinations of (μ_1,μ_2), then f will be as

$$f=1+\mathrm{e}^{\kappa_1}+\mathrm{e}^{\kappa_2}+\mathrm{e}^{A_{12}+\kappa_1+\kappa_2}=1+\mathrm{e}^{\kappa_1}+\mathrm{e}^{\kappa_2}+a_{12}\mathrm{e}^{\kappa_1+\kappa_2},$$

where $a_{12}=\mathrm{e}^{A_{12}}$ is a phase shift coefficient.
For $N=3$, we have $\mu_1,\mu_2,\mu_3=0,1$ so the total eight combinations are
(μ_1,μ_2,μ_3), then f will be as

$$f=1+\mathrm{e}^{\kappa_1}+\mathrm{e}^{\kappa_2}+\mathrm{e}^{\kappa_3}+\mathrm{e}^{A_{12}+\kappa_1+\kappa_2}+\mathrm{e}^{A_{13}+\kappa_1+\kappa_3}+\mathrm{e}^{A_{23}+\kappa_2+\kappa_3}+\mathrm{e}^{A_{12}+A_{13}+A_{23}+\kappa_1+\kappa_2+\kappa_3},$$

or

$$f=1+\mathrm{e}^{\kappa_1}+\mathrm{e}^{\kappa_2}+\mathrm{e}^{\kappa_3}+a_{12}\mathrm{e}^{\kappa_1+\kappa_2}+a_{13}\mathrm{e}^{\kappa_1+\kappa_3}+a_{23}\mathrm{e}^{\kappa_2+\kappa_3}+b_{123}\mathrm{e}^{\kappa_1+\kappa_2+\kappa_3},$$

with $a_{ij}=\mathrm{e}^{A_{ij}}; 1\le i<j\le 3$ and $b_{123}=\mathrm{e}^{A_{12}+A_{13}+A_{23}}=a_{12}a_{13}a_{23}$.

4.1 1-soliton solution

Taking the function f in equation (16) as

$$f_1=f(x,y,z,s,t)=1+\mathrm{e}^{\kappa_1}=1+\mathrm{e}^{g_1x+h_1y+p_1z+q_1s-w_1(t)}, \tag{18}$$

which satisfy the equation (16) completely under the restriction (13). On putting the equations (18) into the equation (14), a single soliton is obtained as

$$u_1=\frac{12g_1^2k\exp\left(\frac{\int\left(g_1^4kA(t)+g_1(h_1C_2(t)+p_1C_3(t)+q_1C_4(t))+g_1^2C_1(t)+h_1^2C_5(t)+p_1^2C_6(t)+q_1^2C_7(t)\right)dt}{g_1}+g_1x+h_1y+p_1z+q_1s\right)}{\left(\exp\left(\frac{\int\left(g_1^4kA(t)+g_1(h_1C_2(t)+p_1C_3(t)+q_1C_4(t))+g_1^2C_1(t)+h_1^2C_5(t)+p_1^2C_6(t)+q_1^2C_7(t)\right)dt}{g_1}\right)+e^{g_1x+h_1y+p_1z+q_1s}\right)^2}. \tag{19}$$

4.2 2-soliton solution

We assume the function f as

$$f_2=f(x,y,z,s,t)=1+\mathrm{e}^{\kappa_1}+\mathrm{e}^{\kappa_2}+a_{12}\mathrm{e}^{\kappa_1+\kappa_2}, \tag{20}$$

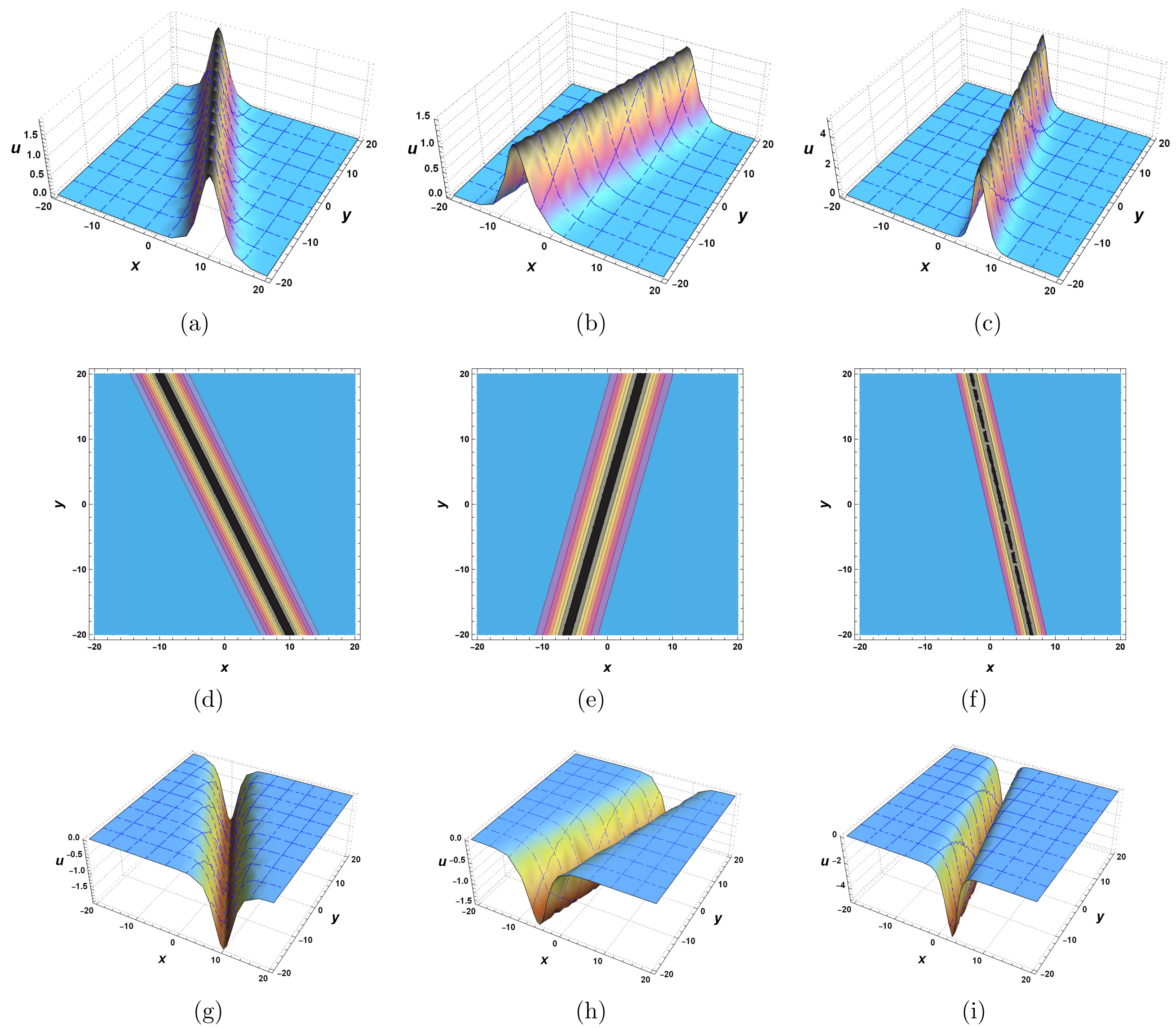


Figure 1: Single bright and dark solitons for (19) having values: $C_i(t) = 1; 2 \le i \le 7, p_1 = q_1 = 1, t = z = s = 0$ with **(a)** $A(t) = t^2, k = 1, C_1(t) = 1, g_1 = 0.8, h_1 = 0.4$; **(b)** $A(t) = \sin(t), k = 1, C_1(t) = 1, g_1 = -0.7, h_1 = 0.2$; **(c)** $A(t) = \exp(t), k = 1, C_1(t) = 2, g_1 = 1.3, h_1 = 0.3$; **(g)** $A(t) = t^2, k = -1, C_1(t) = 1, g_1 = 0.8, h_1 = 0.4$; **(h)** $A(t) = \sin(t), k = -1, C_1(t) = 1, g_1 = -0.7, h_1 = 0.2$; and **(i)** $A(t) = \exp(t), k = -1, C_1(t) = 2, g_1 = 1.3, h_1 = 0.3$. **(d)-(f)** depicts the contour plots for **(a)-(c)** respectively.

where a_{12} is the phase shift coefficient which can be determined by substituting f and its derivatives from equation (20) into the bilinear form (16). Symbolic computations resolve the values of a_{12}, as showcased below:

$$a_{12} = \frac{3kA(t)\,(g_1 - g_2)^{\,2} g_1^2 g_2^2 - C_5(t)\,(g_2 h_1 - g_1 h_2)^{\,2} + C_6(t)\,(g_2 p_1 - g_1 p_2)^{\,2} + C_7(t)\,(g_2 q_1 - g_1 q_2)^{\,2}}{3kA(t)\,(g_1 + g_2)^{\,2} g_1^2 g_2^2 - C_5(t)\,(g_2 h_1 - g_1 h_2)^{\,2} + C_6(t)\,(g_2 p_1 - g_1 p_2)^{\,2} + C_7(t)\,(g_2 q_1 - g_1 q_2)^{\,2}}. \tag{21}$$

Thus, the equation (20) with phase shift (21) satifies the equation (16) under the restriction

$$A(t) = C_5(t)\,(g_2 h_1 - g_1 h_2)^{\,2} + C_6(t)\,(g_2 p_1 - g_1 p_2)^{\,2} + C_7(t)\,(g_2 q_1 - g_1 q_2)^{\,2}, \tag{22}$$

which occurs due to the dependency of phase shift coefficient on the time variable, while on substituting the equation (20) into the equation (16), it was treated as constant coefficient. The substitution of equation (20) with equations (21) and (22) into the equation (14) gives a two-soliton solution for equation (1)

$$u_2(x, y, z, s, t) = 12k(\ln f_2)_{xx}. \tag{23}$$

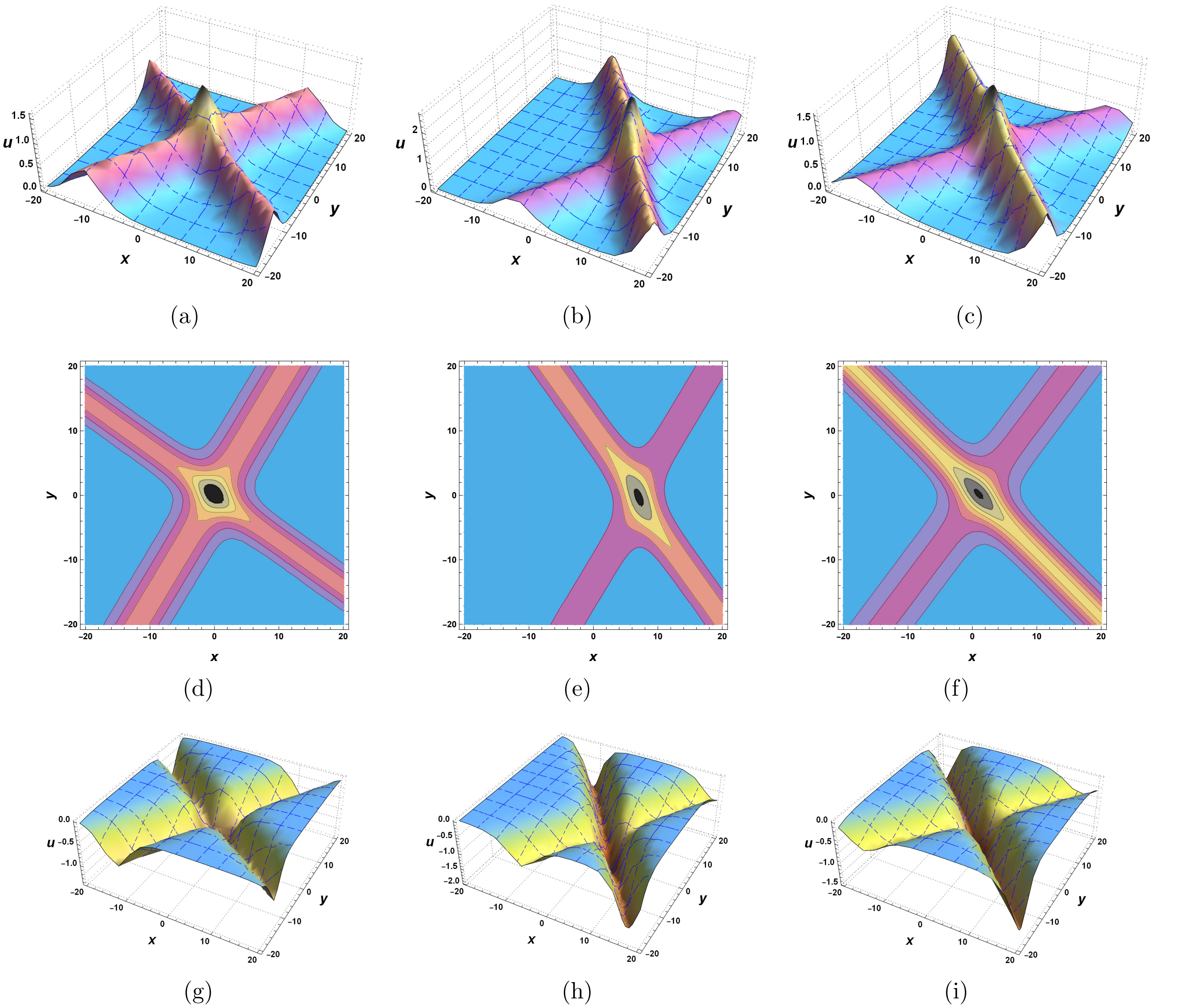


Figure 2: Two bright and dark solitons and their interactions for (23) with (20) having values: $C_i(t) = 1; 2 \leq i \leq 7, q_1 = p_2 = q_2 = 1, t = z = s = 0$ with **(a)** $C_1(t) = t^2, k = 1, g_1 = 0.5, h_1 = 0.7, p_1 = 1, g_2 = -0.5, h_2 = 0.3$; **(b)** $C_1(t) = \sin(t), k = 1, g_1 = 0.7, h_1 = 0.5, p_1 = 1, g_2 = -0.5, h_2 = 0.3$; **(c)** $C_1(t) = \exp(t), k = 1, g_1 = 0.6, h_1 = 0.6, p_1 = 0.5, g_2 = -0.4, h_2 = 0.3$; **(g)** $C_1(t) = t^2, k = -1, g_1 = 0.5, h_1 = 0.7, p_1 = 1, g_2 = -0.5, h_2 = 0.3$; **(h)** $C_1(t) = \sin(t), k = -1, g_1 = 0.7, h_1 = 0.5, p_1 = 1, g_2 = -0.5, h_2 = 0.3,$; and **(i)** $C_1(t) = \exp(t), k = -1, g_1 = 0.6, h_1 = 0.6, p_1 = 0.5, g_2 = -0.4, h_2 = 0.3$. **(d)-(f)** depicts the contour plots for **(a)-(c)** respectively.

4.3 3-soliton solution

For 3-soliton solution, we take function f as

$$f_3 = f(x,y,z,s,t) = 1 + \mathrm{e}^{\kappa_1} + \mathrm{e}^{\kappa_2} + \mathrm{e}^{\kappa_3} + a_{12}\mathrm{e}^{\kappa_1+\kappa_2} + a_{13}\mathrm{e}^{\kappa_1+\kappa_3} + a_{23}\mathrm{e}^{\kappa_2+\kappa_3} + b_{123}\mathrm{e}^{\kappa_1+\kappa_2+\kappa_3}, \tag{24}$$

where a_{ij} with $1 \le i < j \le 3$ are phase shift coefficients, which can be obtained by extrapolating the equation (21) for the auxiliary function

$$f(x,y,z,s,t) = 1 + \mathrm{e}^{\kappa_i} + e^{\kappa_j} + a_{ij}\mathrm{e}^{\kappa_i+\kappa_j}, \tag{25}$$

as

$$a_{ij} = \frac{3kA(t)\,(g_i-g_j)^{\,2} g_i^2 g_j^2 - C_5(t)\,(g_j h_i - g_i h_j)^{\,2} - C_6(t)\,(g_j p_i - g_i p_j)^{\,2} - C_7(t)\,(g_j q_i - g_i q_j)^{\,2}}{3kA(t)\,(g_i+g_j)^{\,2} g_i^2 g_j^2 - C_5(t)\,(g_j h_i - g_i h_j)^{\,2} - C_6(t)\,(g_j p_i - g_i p_j)^{\,2} - C_7(t)\,(g_j q_i - g_i q_j)^{\,2}}, \tag{26}$$

with $1 \le i < j \le N$, where N is an integer. and the coefficient b_{123} can be computed as

$$b_{123} = a_{12}a_{13}a_{23}. \tag{27}$$

Therefore, by substituting the equation (24), with equations (26) and (27) into the equation (14), under the adequate restriction with equation (22), we establish the 3-soliton solution as

$$u_3(x,y,z,s,t) = 12k(\ln f_3)_{xx}. \tag{28}$$

5 Center-controlled N-rogue waves

5.1 Symbolic computational technique

Let Q be a nonlinear PDE of (4+1)-dimensional as

$$Q(u, u_x, u_y, u_z, u_s, u_t, u_{xx}, u_{xt}, u_{xy}, u_{yt}, u_{xz}, u_{zt}, \cdots) = 0, \tag{29}$$

with partial derivatives in independent variables $\{x, y, z, s, t\}$.
We transform the equation (29) with

$$u = u(\xi, \eta), \qquad \xi = k_1 x + k_2 t, \quad \eta = k_3 y + k_4 z + k_5 s, \tag{30}$$

where $k_i; 1 \le i \le 5$ are constants. On putting the equation (30) into equation (29), we get a transformed PDE as

$$T(u, u_\xi, u_{\xi\eta}, u_{\xi\xi}, u_\eta, u_{\eta\eta}, \cdots) = 0, \tag{31}$$

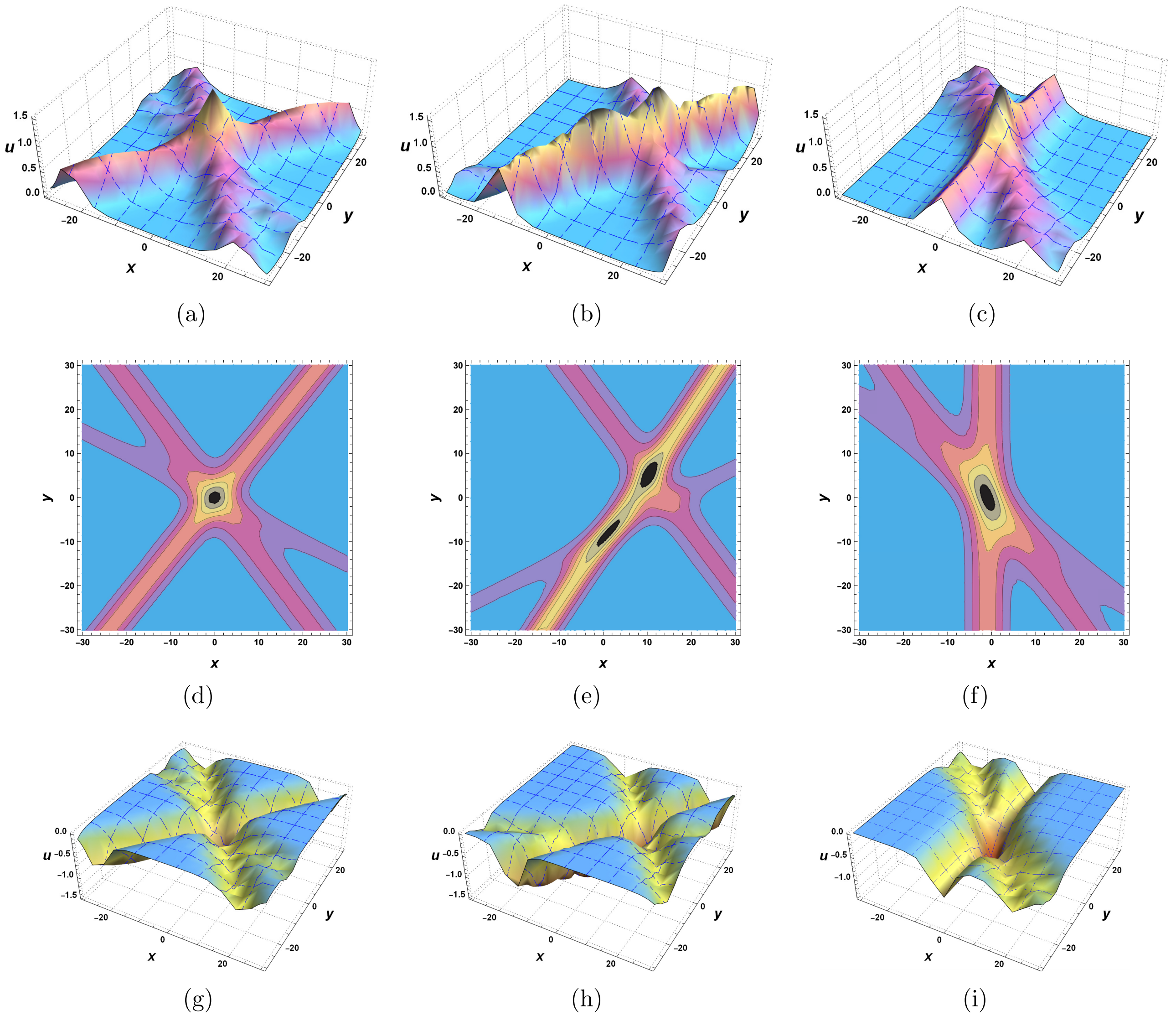
Now, consider a Cole-Hopf transformation as

$$u = P\{\ln f(\xi,\eta)\}_{p\xi}, \tag{32}$$

where P and f are constant and auxiliary function, respectively. Here, p represents the order of ξ, which is a number that balance the highest-order and nonlinear term in the equation (31). On substituting the transformation (32) into (31), we get a bilinear equation in auxiliary function f.
For determining N-rogue waves, we consider a generalized form for f given by Kumar-Mohan [46] with center-parameters β and γ as

$$f = \widehat{f_N}(\xi,\eta,\beta,\gamma) = \sum_{j=0}^{\frac{N}{2}(N+1)} \sum_{i=0}^{j} c_{N(N+1)-2j,2i}(\xi-\beta)^{N(N+1)-2j}(\eta-\gamma)^{2i}, \tag{33}$$

where $c_{m,n}; m, n \in \{0, 2, \cdots, j(j+1)\}$ are the constants.



(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Figure 3: Three bright and dark solitons and their interactions for (28) with (24) having values: $C_i(t) = 1; 2 \leq i \leq 7$ with **(a)** $C_1(t) = t, k = 1, g_1 = 0.3, h_1 = 0.6, p_1 = 0.5, q_1 = 1, g_2 = 0.4, h_2 = 0.3, p_2 = q_2 = 1, g_3 = -0.5, h_3 = 0.4, p_3 = q_3 = 1, t = z = s = 0$; **(b)** $C_1(t) = t^2, k = 1, g_1 = -0.3, h_1 = 0.6, p_1 = 0.5, q_1 = 1, g_2 = 0.4, h_2 = 0.3, p_2 = q_2 = 1, g_3 = -0.6, h_3 = 0.4, p_3 = q_3 = t = z = s = 1$; **(c)** $C_1(t) = \sin(t), k = 1, g_1 = 0.3, h_1 = 0.4, p_1 = 0.5, q_1 = 1.5, g_2 = 0.4, h_2 = 0.3, p_2 = q_2 = 1, g_3 = -0.5, h_3 = 0, p_3 = 0.5, q_3 = -1, t = z = s = 0$; **(g)** $C_1(t) = t, k = -1, g_1 = 0.3, h_1 = 0.6, p_1 = 0.5, q_1 = 1, g_2 = 0.4, h_2 = 0.3, p_2 = q_2 = 1, g_3 = -0.5, h_3 = 0.4, p_3 = q_3 = 1, t = z = s = 0$; **(h)** $C_1(t) = t^2, k = -1, g_1 = -0.3, h_1 = 0.6, p_1 = 0.5, q_1 = 1, g_2 = 0.4, h_2 = 0.3, p_2 = q_2 = 1, g_3 = -0.6, h_3 = 0.4, p_3 = q_3 = t = z = s = 1$; and **(i)** $C_1(t) = \sin(t), k = -1, g_1 = 0.3, h_1 = 0.4, p_1 = 0.5, q_1 = 1.5, g_2 = 0.4, h_2 = 0.3, p_2 = 1, q_2 = 1, g_3 = -0.5, h_3 = 0, p_3 = 0.5, q_3 = -1, t = z = s = 0$. **(d)-(f)** depicts the contour plots for **(a)-(c)** respectively.

5.2 Cole-Hopf transformation and bilinear form

We can rewrite the equation (1) for $C_i = C_i(t)$ as

$$u_{xt} + A(t)\left(uu_x\right)_x + B(t)u_{xxxx} + C_1u_{xx} + C_2u_{xy} + C_3u_{xz} + C_4u_{xs} + C_5u_{yy} + C_6u_{zz} + C_7u_{ss} = 0, \tag{34}$$

We transformed the equation (34) by taking $u = u(\xi, \eta)$, with $\xi = x + t$ and $\eta = y + z + s$ as

$$A(t)(u_{\xi}^2 + uu_{\xi\xi}) + B(t)u_{\xi\xi\xi\xi} + (C_1 + 1)u_{\xi\xi} + (C_2 + C_3 + C_4)u_{\xi\eta} + (C_5 + C_6 + C_7)u_{\eta\eta} = 0, \tag{35}$$

Considering phase variable ϕ_i in equation (35) as

$$\phi_i = k_i \xi - d_i \eta, \tag{36}$$

where k_i are the constants, and d_i are the coefficients of dispersion. On substituting $u = e^{\phi_i}$ in linear terms of equation (35), we obtain d_i as

$$d_i = \frac{(C_2 + C_3 + C_4)p_i \pm \sqrt{p_i^2 \left((C_2 + C_3 + C_4)^2 - 4\,(C_5 + C_6 + C_7)\left(B(t)p_i^2 + C_1 + 1\right)\right)}}{2\,(C_5 + C_6 + C_7)}. \tag{37}$$

Next, taking the Cole-Hopf transformation as

$$u = P(\log f)_{\xi\xi}, \tag{38}$$

and substitute it with $f = 1 + e^{\phi_1}$ and equation (37) into (35). We get the value of K as

$$P = \frac{12B(t)}{A(t)}.$$

The transformation (38) transforms the equation (35) into a bilinear equation as

$$\begin{aligned} B(t)(3f_{\xi\xi}^2 - 4f_{\xi}f_{\xi\xi\xi} + ff_{\xi\xi\xi\xi}) + (C_1 + 1)(ff_{\xi\xi} - f_{\xi}^2) + (C_2 + C_3 + C_4)(ff_{\xi\eta} - f_{\xi}f_{\eta}) \\ + (C_5 + C_6 + C_7)(ff_{\eta\eta} - f_{\eta}^2) = 0, \end{aligned} \tag{39}$$

and gives Hirota's D-operator bilinear form as

$$\left[B(t)D_{\xi}^4 + (C_1 + 1)D_{\xi}^2 + (C_2 + C_3 + C_4)D_{\xi}D_{\eta} + (C_5 + C_6 + C_7)D_{\eta}^2\right] f.f = 0. \tag{40}$$

5.3 First-order rogue waves

For $N = 1$ in equation (33), we consider f as

$$f = c_{2,0}\xi^2 + c_{0,2}\eta^2 + c_{0,0}. \tag{41}$$

Substituting the equation (41) into the equation (39), and equating coefficients to zero for different powers of $\xi^m \eta^n; m, n \in Z$, gives a system of equations as

$$\begin{aligned} 2c_{0,2}\left(c_{2,0}\,(C_1 + 1) - c_{0,2}\,(C_5 + C_6 + C_7)\right) &= 0, \\ 12c_{2,0}^2 B(t) + 2c_{0,0}\left(c_{2,0}\,(C_1 + 1) + c_{0,2}\,(C_5 + C_6 + C_7)\right) &= 0, \end{aligned} \tag{42}$$

which gives the solution for constants as

$$c_{0,2} = \frac{(C_1 + 1)\,c_{2,0}}{C_5 + C_6 + C_7}, \qquad c_{0,0} = -\frac{3B(t)c_{2,0}}{C_1 + 1}, \qquad c_{2,0} = c_{2,0}. \tag{43}$$

So, with the values in equation (43), the equation (41) will be a solution of equation (39) as

$$f = \widehat{f}_1(\xi, \eta, \beta, \gamma) = \left(-\frac{3B(t)}{C_1 + 1} + (\beta - \xi)^2 + \frac{(\gamma - \eta)^2\,(C_1 + 1)}{C_5 + C_6 + C_7}\right) c_{2,0}. \tag{44}$$

Thus, we get a first-order rogue wave solution by substituting equation (44) into the equation (38) as

$$u(\xi, \eta) = \frac{24B(t)\left(-\frac{3B(t)}{C_1+1} - (\beta - \xi)^2 + \frac{(\gamma-\eta)^2(C_1+1)}{C_5+C_6+C_7}\right)}{A(t)\left(-\frac{3B(t)}{C_1+1} + (\beta - \xi)^2 + \frac{(\gamma-\eta)^2(C_1+1)}{C_5+C_6+C_7}\right)^2}, \tag{45}$$

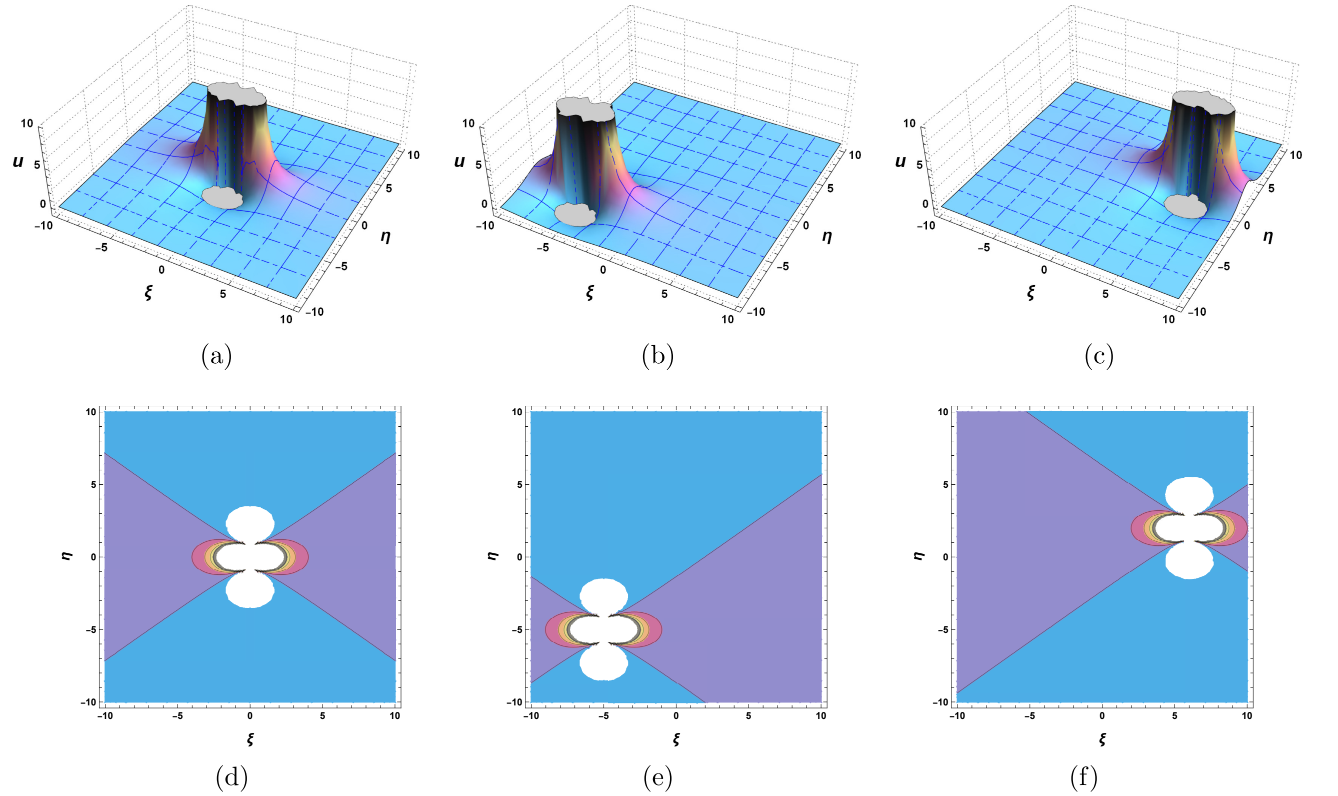


Figure 4: Single rogue waves for (45) with values: $A(t) = -t^2, B(t) = C_1(t) = C_5(t) = 1, C_6(t) = C_7(t) = 0$, and (β, γ) as **(a)** $(0,0)$; **(b)** $(-5,-5)$; and **(c)** $(6,2)$. **(d-f)** shows contours for **(a-c)** in $\xi\eta$-plane.

5.4 Rogue waves of second-order

For $N = 2$ in equation (33), we consider f as

$$f = c_{6,0}\xi^6 + c_{4,2}\eta^2\xi^4 + c_{4,0}\xi^4 + c_{2,4}\eta^4\xi^2 + c_{2,2}\eta^2\xi^2 + c_{2,0}\xi^2 + c_{0,6}\eta^6 + c_{0,4}\eta^4 + c_{0,2}\eta^2 + c_{0,0}. \tag{46}$$

Substituting the equation (46) into the equation (39), and equating coefficients to zero for different powers of $\xi^m\eta^n; m, n \in Z$, we get a system. This system gives constants as

$$c_{0,0} = -\frac{7625B(t)^3\,(C_5 + C_6 + C_7)\,c_{4,2}}{64\,(C_1 + 1)^4}, c_{0,2} = -\frac{13175B(t)^2c_{4,2}}{192\,(C_1 + 1)^2}, c_{0,4} = -\frac{143B(t)c_{4,2}}{12\,(C_5 + C_6 + C_7)},$$

$$c_{0,6} = \frac{(C_1 + 1)^2c_{4,2}}{3\,(C_5 + C_6 + C_7)^2}, c_{2,0} = -\frac{1175B(t)^2\,(C_5 + C_6 + C_7)\,c_{4,2}}{192\,(C_1 + 1)^3}, c_{2,2} = -\frac{30B(t)c_{4,2}}{C_1 + 1},$$

$$c_{2,4} = \frac{(C_1 + 1)\,c_{4,2}}{C_5 + C_6 + C_7}, c_{4,0} = -\frac{25B(t)\,(C_5 + C_6 + C_7)\,c_{4,2}}{12\,(C_1 + 1)^2}, c_{4,2} = c_{4,2}$$

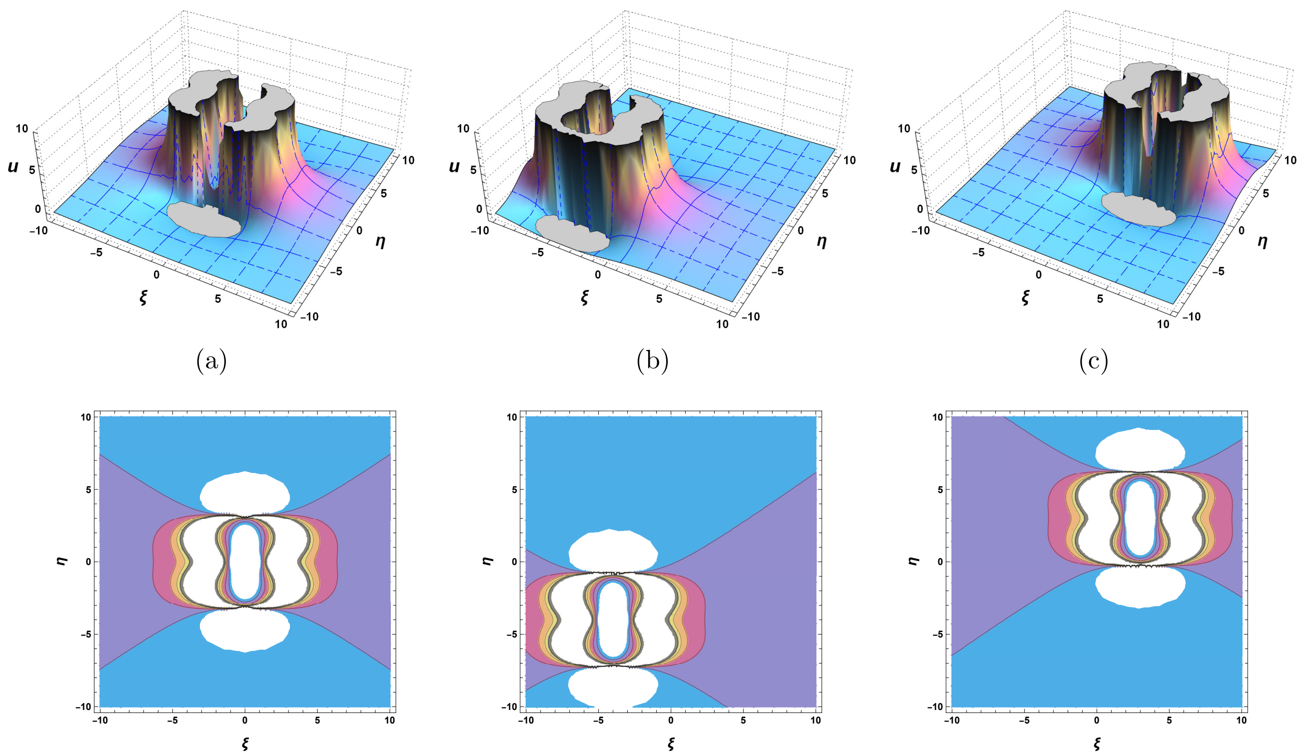
$$c_{6,0} = \frac{(C_5 + C_6 + C_7)\,c_{4,2}}{3\,(C_1 + 1)}, \tag{47}$$

So, with the values in equation (47), the equation (46) will be a solution of equation (39) as

$$
\begin{aligned}
f = \widehat{f}_2(\xi,\eta,\beta,\gamma) = \frac{c_{4,2}}{192}(192(\beta-\xi)^4(\gamma-\eta)^2 - \frac{5760(\beta-\xi)^2B(t)(\gamma-\eta)^2}{C_1+1} \\
- \frac{400(\beta-\xi)^4B(t)\,(C_5+C_6+C_7)}{(C_1+1)^{\,2}} - \frac{1175(\beta-\xi)^2B(t)^2\,(C_5+C_6+C_7)}{(C_1+1)^{\,3}} \\
- \frac{2288B(t)(\gamma-\eta)^4}{C_5+C_6+C_7} - \frac{13175B(t)^2(\gamma-\eta)^2}{(C_1+1)^{\,2}} - \frac{22875B(t)^3\,(C_5+C_6+C_7)}{(C_1+1)^{\,4}} \\
+ \frac{192(\beta-\xi)^2(\gamma-\eta)^4\,(C_1+1)}{C_5+C_6+C_7} + \frac{64(\beta-\xi)^6\,(C_5+C_6+C_7)}{C_1+1} + \frac{64(\gamma-\eta)^6\,(C_1+1)^{\,2}}{(C_5+C_6+C_7)^{\,2}}).
\end{aligned}
\tag{48}
$$

Thus, a solution for second-order rogue wave is obtained by substituting equation (48) into the equation (38) as

$$
u(\xi,\eta) = \frac{12B(t)}{A(t)}(\log\widehat{f}_2)_{\xi\xi}. \tag{49}
$$



(a) (b) (c)

(d) (e) (f)

Figure 5: Second-order rogue waves for (49) with values: $A(t) = -t^2, B(t) = C_1(t) = C_5(t) = 1, C_6(t) = C_7(t) = 0$, and and (β,γ) as **(a)** $(0,0)$; **(b)** $(-4,-4)$; and **(c)** $(3,3)$. **(d-f)** shows contours for **(a-c)** in $\xi\eta$-plane.

6 Results and findings

The investigated generalized equation (1) shows complete integrability so that it can have several solutions, such as lumps, breathers, kinks, and others. With the appropriate choice of parameters, it established the solutions for solitons up to third order using the Hirota bilinear technique (HBT). Due to the dependency of the phase shift coefficient on time variable t in HBT, we found the restriction for these soliton solutions. This work studied rogue wave solutions up to second order using symbolic computational techniques and demonstrated the dynamic structures of these solutions. In this work, due to the integrablility of this KdV-type equation, solitons follow the strict behavior of their property that they do not change their shapes with respect to the time 't,' and only can be seen shifted at the time of phase shift as we have seen in the 3D graphics. Since the change in time variable 't,' will only shift the dynamics to the moving direction, we have not included the 2D plots for the same as this is not required in our case. We showed the dynamic analysis of the solutions obtained with different constant and variable parameters for all graphics, and analyzed the interaction of solitons concerning different parameters and with variable coefficients such as $\sin(t), t^2$, and $\exp(t)$ for graphics at different stages. The impact of the variable coefficients on the solutions can be seen in their steepness, amplitude, interaction points, and behaviors. Similarly, for the study of rogue waves, the solution behaviors with respect to the center-controlled parameters having different constant parameters and variable coefficients were shown. This study explains the findings as follows:

- Figure-1 shows the single bright solitons in (a)-(c) and the single dark solitons in (g)-(i) due to the moving singularities around the x-axis. Graphics in (d)-(f) illustrate the contour plots and give the observation of moving singularities as near to $x = 10$, -6, and 8 for (a)-(c), respectively.

- In figure-2, we depict the soliton interactions of two bright-solitons in (a)-(c) and of two dark-solitons in (g)-(i) due to the moving singularities around the x-axis. Graphics in (d)-(f) illustrate the contour plots and observe moving singularities—the interaction of the solitons, showing the X-type interactions for all the graphics.

- Figure-3 illustrates the soliton interactions of three bright-solitons in (a)-(c) and of three dark-solitons in (g)-(i) due to the moving singularities around the x-axis. Graphics in (d)-(f) illustrate the contour plots and observe moving singularities. We observe that the interaction of the solitons in (d)-(f) occurs at one point only, while the interaction of the solitons in (e) occurs at two points for (a)-(b) and (c), respectively.

- In figure-4, we show the rogue waves of first-order, which shows the dynamics of single rogues in (a)-(c) due to the singularity at center-parameters β and γ. Graphics in (d)-(f) illustrate the contour plots and give the observation of rogue waves with moving singularities at center-controlled parameters.

- Figure-5 shows the rogue waves second-order in (a)-(c) and illustrates the interaction of two rogues due to singularity corresponding to the center-parameters β and γ. Graphics in (d)-(f) illustrate the contour plots and give the observation of rogue waves with moving singularities due to center-controlled parameters.

7 Conclusions

This research investigated a $(4 + 1)$-dimensional variable coefficient generalized KP equation in fluid mechanics. It analyzed the complete integrability using the robust tool Painlevé analysis of this generalized equation, with arbitrary choices and fulfillment of conditions for compatibility for the obtained resonances. Using the Cole-Hopf transformation, it founds the bilinear equation and obtained its Hirota's bilinear form in the auxiliary function of the studied equation using the bilinear differential operator. It obtained the bright and dark solitons up to third order and their one- or multiple-point interactions by choosing appropriate values for the parameters for N-soliton solutions under the restrictions. These restrictions occur due to the dependency of the phase shift coefficient on time t. After that, rogue wave solutions were studied up to the second order using the symbolic computational technique for N-rogue waves with appropriate values for the parameters. It showed the dynamic structures of these solutions with several chosen parameters.

Due to the generalized nature of the investigated equation with variable coefficients, it has applications in fluid dynamics, the theory of solitons, nonlinear waves, plasmas, and other nonlinear sciences. Thus, this equation has a broad scope for investigating different water wave solutions such as lumps, kinks, and breathers. We have employed the Hirota bilinear and symbolic computational techniques to obtain the solitons and rogue waves. Researchers and scientists can investigate this equation using techniques and methods such as Lie symmetry analysis, Darboux transformation, and other methodologies.

Declarations

Ethics approval and consent to participate

Not applicable.

Competing interests

The authors state that there is no conflict of interest.

Authors' contributions

Both authors have agreed and given their consent for the publication of this research paper.

Data availability statement

No data is used from outer sources in this work.

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