A novel and efficient method for obtaining Hirota's bilinear form for the nonlinear evolution equation in (n+1) dimensions

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Abstract

Bilinearization of nonlinear partial differential equations (PDEs) is essential in the Hirota method, which is a widely used and robust mathematical tool for finding soliton solutions of nonlinear PDEs in a variety of fields, including nonlinear dynamics, mathematical physics, and engineering sciences. We present a novel systematic computational approach for determining the bilinear form of a class of nonlinear PDEs in this article. It can be easily implemented in symbolic system software like Mathematica, Matlab, and Maple because of its simplicity. The proven results are obtained by using a developed method in Mathematica and applying a logarithmic transformation to the dependant variable. Finally, the findings validate the implemented technique's competence, productivity, and dependability. The approach is a useful, authentic, and simple mathematical tool for calculating multiple soliton solutions to nonlinear evolution equations encountered in nonlinear sciences, plasma physics, ocean engineering, applied mathematics, and fluid dynamics.

Keywords: Hirota method; Bilinear form; Dependent variable transformation; Nonlinear evolution equations; Logarithmic transformation; Symbolic computation.
MSC: 39A14; 33F10; 35C05; 35C07; 35C09.

1. Introduction

The Hirota method, which is a widely used and robust mathematical tool for finding soliton solutions of nonlinear partial differential equations(PDEs) in a range of domains such as nonlinear dynamics, mathematical physics, oceanography, engineering sciences, and others requires bilinearization of nonlinear PDEs. By taking the advantage of Hirota's bilinear method[1–8], one can obtain the exact N-soliton or multi-soliton solutions for integrable nonlinear PDEs, but the important step in this method is the transformation of a nonlinear PDE into the bilinear form as deduced by Hirota[9]. The conversion of a nonlinear PDE into the bilinear form becomes tedious even if the corresponding dependent variable transformation is known. Therefore, the development of an algorithm to get the bilinear form for a nonlinear PDE plays an important role, and the utilization of computer algebra system software such as *Mathematica, Matlab* and *Maple* can be productive in carrying out such computations.

The nonlinearity of partial differential equations[10–27] have captured the attention of numerous researchers, where they have made use of several methodical approaches to achieve multiple solitons, breather, and lump solutions. Many approaches other than the Hirota method have been brought into

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practice to determine the exact solutions for the nonlinear PDEs, such as Darboux transformation, simplified Hirota method, Bcklund transformation, Lie symmetry analysis, Pfaffian technique, Inverse scattering method, and several other methods.

To establish the exact N-soliton solutions of the Kortewegde Vries (KdV), sineGordon (SG), modified KdV, and nonlinear Schrdinger equations, a direct method was developed by Hirota[28–30] and Hietarinta[31–34] by making use of Hirota's 3-soliton condition for nonlinear PDEs and carried out a search for the bilinear equations for KdV-type, SG-type, modified KdV-type, and nonlinear Schrdinger-type equations. In 1995, Hereman and Zhuang[35] summarized the types of bilinear equations based on the work done by Hietarinta.

Zhou, Fu, and Li[36,37] gave an algorithm to construct the bilinear forms of KdV type nonlinear PDEs by making use of the recursive form of D-operator using logarithmic transformation. Although their algorithm established a generalized bilinear form up to (2+1)-dimension, it fails to generalized the (n+1)-dimensional bilinear form. Our algorithm does not utilize the recursive form of the D-operator, but instead, it employs the expression for the D-operator as derived by Hereman[35]. Therefore, the established algorithm becomes more efficient when compared with the algorithm of Zhou, Fu, and Li. Also, the prime objective of this article is to generalize the bilinear form for (n+1)-dimensional nonlinear PDEs by applying a logarithmic transformation to the dependent variable. The established results of the demonstrated examples in section-4 have been exhibited through implementation and execution of the algorithm in the computational software *Mathematica*.

This research work is arranged as follows: In the forthcoming section, the Hirota's bilinear form for Sawada-Kotera nonlinear PDE as an example is reviewed. In Section 3, a new and efficient algorithm for constructing the bilinear forms of nonlinear PDEs is established. Section-4 illustrates different examples of well-known nonlinear PDEs such as KdV equation, KP equation, SK equation, shallow water wave equation, BLMP equation, HSI system, generalized BKP equation, Fokas equation, and others from the fields of nonlinear dynamics, mathematical physics, oceanography, plasma physics, and other sciences using the system software Mathematica. Section-6 concludes our research work.

2. Hirota's Bilinear form

Hirota gave an algebraic method to find exact soliton solutions to nonlinear PDEs, provided the NLPDE can be transformed into bilinear form. The foremost step of the method was changing the nonlinear PDE to a quadratic or quartic equation for an auxiliary function using dependent variable transformation. We can understand it with an example of the Sawada-Kotera equation[42]

$$u_t + 5(uu_{2x})_x + 5u^2u_x + u_{5x} = 0, \tag{2.1}$$

where u is a function of x and t. Considering dependent variable transformation

$$u = 6(ln\Phi)_{2x}, \tag{2.2}$$

where $\Phi = \Phi(x,t)$. Substituting (2.2) into (2.1) and integrating once w.r.t. x by choosing constant of integration as zero, we get the following

$$\Phi\Phi_{xt} - \Phi_x\Phi_t + 15\Phi_{2x}\Phi_{4x} - 10\Phi_{3x}^2 + \Phi\Phi_{6x} - 6\Phi_x\Phi_{5x} = 0, \tag{2.3}$$

which is a quadratic equation in Φ. Hirota defined the D-operators in[9] as

$$D_x^n\Phi.\Psi = \left(\frac{\partial}{\partial_{x_1}} - \frac{\partial}{\partial_{x_2}}\right)^n \Phi(x_1)\Psi(x_2)|_{x_1=x_2=x}, \tag{2.4}$$

$$D_x^n D_t^m \Phi.\Psi = \left(\frac{\partial}{\partial_{x_1}} - \frac{\partial}{\partial_{x_2}}\right)^n \left(\frac{\partial}{\partial_{t_1}} - \frac{\partial}{\partial_{t_2}}\right)^m \Phi(x_1,t_1)\Psi(x_2,t_2)|_{x_1=x_2=x,t_1=t_2=t}. \tag{2.5}$$

Using the D-operators from equations (2.4) and (2.5), we can rewrite the equation (2.3) in bilinear form as

$$(D_x D_t + D_x^6)\Phi.\Phi = 0. \tag{2.6}$$

3. Algorithm

We start by considering the (n+1)-dimensional nonlinear PDE

$$P(u, u_{x_1}, u_{x_2}, \cdots) = 0, \tag{3.1}$$

which contains $u = u(x_1, x_2, ..., x_n, t)$ and its partial derivatives with respect to the independent variables $x_1, x_2, ..., x_n$ and t.

Step 1: Finding the dispersion

We consider the phase variable Θ_i as

$$\Theta_i = k_{1_i}x_1 + k_{2_i}x_2 + k_{3_i}x_3 + \cdots + k_{n_i}x_n + w_i t, \tag{3.2}$$

where $k_{N_i}; 1 \leq N \leq n$ are constants and w_i is the dispersion. We have chosen the standard phase variable relation to show the applicability of the algorithm for different examples, discussed in Section 4, but consideration of it may vary depending upon the structure of a nonlinear PDE. By substituting $u = e^{\Theta_i}$, in the linear terms of (3.1) and solving it for w_i, we get the dispersion.

Step 2: Finding the constant R in dependent variable transformation

We consider the logarithmic transformation as

$$u = R\frac{\partial^\eta}{\partial_{x^\eta}}(ln\Phi), \tag{3.3}$$

where η can be calculated by balancing nonlinear terms, and the highest order derivative in equation (3.1). Considering the function $\Phi = 1 + e^{\Theta_1}$, and substituting in (3.3), we get a set of values for R, from which we can choose appropriate value for it.

Step 3: Finding quadratic or quartic equation in Φ

Substituting the equation (3.3) into the equation (3.1), and integrating w.r.t. x one or more times by choosing integrating constant as zero upto minimum possible order of the equation, we get quadratic or quartic equation in Φ

$$Q(\Phi, \Phi_{x_1}, \Phi_{x_2}, \cdots) = 0, \tag{3.4}$$

which contains Φ and its partial derivatives with respect to the independent variables $x_1, x_2, ..., x_n$, and t.

Step 4: Constructing D-operator and generalized bilinear form of a nonlinear PDE

We define Hirota D-operator (HDO) designed by Hereman[35] as

$$HDO[m,n](p,q) = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{m+n-j-i}\binom{m}{i}\binom{n}{j}\frac{\partial^{i+j}}{\partial q^i \partial p^j}\Phi\frac{\partial^{m+n-i-j}}{\partial q^{n-i}\partial p^{m-j}}\Psi, \tag{3.5}$$

	(1+1)-dim	***(2+1)-dim***	***(3+1)-dim***	***(n+1)-dim***
independent variables	x, t	x_1, x_2, t	x_1, x_2, x_3, t	$x_1, x_2, ..., x_n, t$
number of pairs	$\binom{2}{2} = 1$	$\binom{3}{2} = 3$	$\binom{4}{2} = 6$	$\binom{n}{2} = \frac{n(n-1)}{2}$
pairs	(x, t)	$(x_1, t), (x_1, x_2), (x_2, t)$	$(x_1, t), (x_1, x_2), (x_1, x_3), (x_2, t), (x_2, x_3), (x_3, t)$	$(x_1, t), (x_1, x_2), ... (x_1, x_n), (x_2, t), (x_2, x_3), ..., (x_{n-1}, x_n), (x_n, t)$
generalized bilinear form	$T_a[M, N](x, t)$	$T_a[M, N](x_1, t) + T_b[M, O](x_1, x_2) + T_c[N, O](x_2, t)$	$\sum_{i=1}^{6} T_{C_i}[U_i, V_i](pair_i)$	$\sum_{i=1}^{\frac{n(n-1)}{2}} T_{C_i}[U_i, V_i](pair_i)$

Table 1: Formulation of generalized bilinear form for (n+1)-dimensional nonlinear PDEs.

where p and q are the independent variables and m and n are constants.
Defining a term T as

$$T_C[M, N](p, q) = \sum_{m=0}^{M} \sum_{n=0}^{N} C_{mn} HDO[m, n](p, q), \tag{3.6}$$

where $C : C_{mn}$ are constants, M and N are the highest order in the equation (3.4) for the independent variables p and q respectively. Then we construct generalized bilear form as
(1+1)-dimensional : $T_a[M, N](x, t)$,
(2+1)-dimensional : $T_a[M, N](x_1, t) + T_b[M, O](x_1, x_2) + T_c[N, O](x_2, t)$,
(3+1)-dimensional :
$T_a[M, N](x_1, t) + T_b[M, O](x_1, x_2) + T_c[M, P](x_1, x_3) + T_d[N, O](x_2, t) + T_e[N, P](x_2, x_3) + T_f[O, P](x_3, t)$,
(4+1)-dimensional :
$T_a[M, N](x_1, t) + T_b[M, O](x_1, x_2) + T_c[M, P](x_1, x_3) + T_d[M, Q](x_1, x_4) + T_e[N, O](x_2, t) + T_f[N, P](x_2, x_3) + T_g[N, Q](x_2, x_4) + T_h[O, P](x_3, t) + T_p[O, Q](x_3, x_4) + T_q[P, Q](x_4, t)$,

$$\vdots$$

(n+1)-dimensional :

$$\sum_{i=1}^{\frac{n(n-1)}{2}} T_{C_i}[U_i, V_i](pair_i), \tag{3.7}$$

where C_i are constants, $pair_i$ is the i^{th} pair of independent variables described in Table 1, U_i and V_i are the highest order in the equation (3.4) for independent variables in the $pair_i$.
Step 5: Finding constants in generalized bilinear form
On equating generalized bilinear form (3.7) to the left-hand side of equation (3.4), we get a system of equation in coefficients $C_{i_{mn}}; 1 \le m \le M, 1 \le n \le N$ (if we find the equation (3.4) as quartic then first, we make the equation (3.7) as quartic by multiplication of Φ^2, and then equate). After solving this system and utilizing the constants and relations in the equation (3.7), we get the desired bilinear form for the nonlinear PDE (3.1).

4. Examples

4.1. (1+1)-dimensional equations

4.1.1. Korteweg-de Varies (KdV) equation

Considering the KdV equation[9] as

$$u_t + 6uu_x + u_{3x} = 0. \tag{4.1}$$

Step 1: Finds dispersion $w_i = -\alpha_i^3$ with $u = e^{\alpha_i x + w_i t}$.
Step 2: Finds $R = 2$ for the logarithmic transformation $u = R(ln\Phi)_{xx}$.
Step 3: Finds quadratic equation in Φ as

$$-\Phi_t\Phi_x + \Phi\Phi_{xt} + 3\Phi_{2x}^2 - 4\Phi_x\Phi_{3x} + \Phi\Phi_{4x} = 0. \tag{4.2}$$

Step 4: Constructs D-operator and generalized bilinear form as (3.5) and (3.7) respectively.(This step will be skipped in further examples as this construction will be formulated as same as this).
Step 5: Finds nonzero coefficients $a_{40} = \frac{1}{2}$ and $a_{11} = \frac{1}{2}$. Hence, we constructs bilinear form for equation (4.1) as

$$(D_tD_x + D_x^4)\Phi.\Phi = 0. \tag{4.3}$$

4.1.2. Caudrey-Dodd-Gibbon (CDG) equation

Taking the CDG equation[39] as

$$u_t + u_{5x} + 30uu_{3x} + 30u_xu_{2x} + 180u^2u_x = 0. \tag{4.4}$$

Step 1: Finds dispersion $w_i = -\alpha_i^5$ with $u = e^{\alpha_i x + w_i t}$.
Step 2: Finds $R = 1$ for the transformation $u = R(ln\Phi)_{xx}$.
Step 3: Finds quadratic equation in Φ as

$$-\Phi_t\Phi_x + \Phi\Phi_{xt} - 10\Phi_{3x}^2 + 15\Phi_{2x}\Phi_{4x} - 6\Phi_x\Phi_{5x} + \Phi\Phi_{6x} = 0. \tag{4.5}$$

Step 4: Finds nonzero coefficients $a_{60} = \frac{1}{2}$ and $a_{11} = \frac{1}{2}$. Hence, we constructs bilinear form for equation (4.4) as

$$(D_tD_x + D_x^6)\Phi.\Phi = 0. \tag{4.6}$$

4.1.3. Shallow water wave (SWW) equation

We consider the SWW equation[40] as

$$u_t - u_{(2x)t} - 3uu_t + 3u_x\int_x^{\infty} u_t dx' + u_x = 0. \tag{4.7}$$

Step 1: Finds dispersion $w_i = \frac{\alpha_i}{-1+\alpha_i^2}$ with $u = e^{\alpha_i x + w_i t}$.
Step 2: Finds $R = 2$ for $0 < \alpha_i < 1$ or $\alpha_i > 1$ in $u = R(ln\Phi)_{xx}$.
Step 3: Finds quadratic equation in Φ (not showing due to lengthy expression).
Step 4: Finds nonzero coefficients $a_{11} = 1$, $a_{20} = 1$ and $a_{31} = -1$. Hence, we constructs bilinear form for equation (4.7) as

$$(D_xD_t - D_x^3D_t + D_x^2)\Phi.\Phi = 0. \tag{4.8}$$

4.1.4. Boussinesq equation

Considering the Boussinesq equation[41] as

$$u_{2t} - u_{2x} - 6u_x^2 - 6uu_{2x} - u_{4x} = 0. \tag{4.9}$$

Step 1: Finds dispersion $w_i = \alpha_i\sqrt{1+\alpha_i^2}$ with $u = e^{\alpha_i x + w_i t}$.
Step 2: Finds $R = 2$ for the transformation $u = R(ln\Phi)_{xx}$.
Step 3: Finds quadratic equation in Φ as

$$-\Phi_t^2 + \Phi\Phi_{2t} + \Phi_x^2 - \Phi\Phi_{2x} - 3\Phi_{2x}^2 + 4\Phi_x\Phi_{3x} - \Phi\Phi_{4x} = 0. \tag{4.10}$$

Step 4: Finds nonzero coefficients $a_{02} = \frac{1}{2}$, $a_{20} = -\frac{1}{2}$ and $a_{40} = -\frac{1}{2}$. Hence, we constructs bilinear form for equation (4.9) as

$$(D_t^2 - D_x^2 - D_x^4)\Phi.\Phi = 0. \tag{4.11}$$

4.1.5. Sawada-Kotera (SK) equation

Taking the SK equation[42] as

$$u_t + 5(uu_{2x})_x + 5u^2u_x + u_{5x} = 0. \tag{4.12}$$

Step 1: Finds dispersion $w_i = -\alpha_i^5$ with $u = e^{\alpha_i x + w_i t}$.
Step 2: Finds $R = 6$ for the transformation $u = R(ln\Phi)_{xx}$.
Step 3: Finds quadratic equation in Φ as

$$-\Phi_t\Phi_x + \Phi\Phi_{xt} - 10\Phi_{3x}^2 + 15\Phi_{2x}\Phi_{4x} - 6\Phi_x\Phi_{5x} + \Phi\Phi_{6x} = 0. \tag{4.13}$$

Step 4: Finds nonzero coefficients $a_{11} = \frac{1}{2}$ and $a_{60} = \frac{1}{2}$. Hence, we constructs bilinear form for equation (4.12) as

$$(D_xD_t + D_x^6)\Phi.\Phi = 0. \tag{4.14}$$

4.2. (2+1)-dimensional equations

4.2.1. KadomtsevPetviashvili (KP) equation

Considering the KP equation[43] as

$$(u_t - \frac{1}{4}u_{3x} - 3uu_x)_x - \frac{3}{4}u_{2y} = 0. \tag{4.15}$$

Step 1: Finds dispersion $w_i = \frac{\alpha_i^4 + 3\beta_i^2}{4\alpha_i}$ with $u = e^{\alpha_i x + \beta_i y + w_i t}$.
Step 2: Finds $R = 1$ for the transformation $u = R(ln\Phi)_{xx}$.
Step 3: Finds quadratic equation in Φ as

$$3\Phi_y^2 - 3\Phi\Phi_{2y} - 4\Phi_t\Phi_x + 4\Phi\Phi_{xt} - 3\Phi_{2x}^2 + 4\Phi_x\Phi_{3x} - \Phi\Phi_{4x} = 0. \tag{4.16}$$

Step 4: Finds nonzero coefficients and relations $a_{11} = 2$, $a_{40} + b_{40} = -\frac{1}{2}$ and $b_{02} + c_{20} = -\frac{3}{2}$. Hence, we constructs bilinear form for equation (4.15) as

$$(-D_x^4 - 3D_y^2 + 4D_xD_t)\Phi.\Phi = 0. \tag{4.17}$$

4.2.2. Hirota-Satsuma-Ito (HSI) system

Taking the HSI equation[44] as

$$v_t + u_{(2x)t} + 3(uw)_x + u_x = 0, u_y = v_x, u_t = w_x, \tag{4.18}$$

which can be converted into nonlinear PDE as

$$u_{yt} + u_{(3x)t} + 6u_x u_t + 3u_{xx}\int u_t dx' + 3uu_x + u_{xx} = 0. \tag{4.19}$$

Step 1: Finds dispersion $w_i(t) = -\frac{\alpha_i^2}{\alpha_i^3+\beta_i}$ with $u = e^{\alpha_i x+\beta_i y - w_i(t)}$.
Step 2: Finds $R = 2$ for the transformation $u = R(ln\Phi)_{xx}$.
Step 3: Finds quartic equation in Φ as

$$-\Phi^2\Phi_t\Phi_y + \Phi^3\Phi_{yt} - \Phi^2\Phi_x^2 - 12\Phi\Phi_t\Phi_x^2 - 6\Phi_t\Phi_x^3 + 6\Phi^2\Phi_x\Phi_{xt} + 6\Phi\Phi_x^2\Phi_{xt} + \Phi^3\Phi_{2x} + 6\Phi^2\Phi_t\Phi_{2x} +$$

$$6\Phi\Phi_t\Phi_x\Phi_{2x} - 3\Phi^2\Phi_{xt}\Phi_{2x} - 3\Phi^2\Phi_x\Phi_{(2x)t} - \Phi^2\Phi_t\Phi_{3x} + \Phi^3\Phi_{(3x)t} = 0. \tag{4.20}$$

Step 4: Finds nonzero coefficients and relations $a_{31} = \frac{1}{2}$, $a_{20}+b_{20} = \frac{1}{2}$ and $c_{11} = \frac{1}{2}$. Hence, we constructs bilinear form for equation (4.19) as

$$(D_yD_t + D_x^2 + D_x^3D_t)\Phi.\Phi = 0. \tag{4.21}$$

4.2.3. KP equation with variable coefficient

Considering the KP equation with time-variable coefficient[45] as

$$(u_t + uu_x + u_{3x})_x + 3u_{2y} + F(t)u_{xy} = 0. \tag{4.22}$$

Step 1: Finds dispersion $w_i(t) = \int \frac{\alpha_i^4+F(t)\alpha_i\beta_i+3\beta_i^2}{\alpha_i}dt$ with $u = e^{\alpha_i x+\beta_i y - w_i(t)}$.
Step 2: Finds $R = 12$ for the transformation $u = R(ln\Phi)_{xx}$.
Step 3: Finds quadratic equation in Φ as

$$-3\Phi_y^2 + 3\Phi\Phi_{2y} - \Phi_t\Phi_x - F(t)\Phi_y\Phi_x + \Phi\Phi_{xt} + F(t)\Phi\Phi_{xy} + 3\Phi_{2x}^2 - 4\Phi_x\Phi_{3x} + \Phi\Phi_{4x} = 0. \tag{4.23}$$

Step 4: Finds nonzero coefficients and relations $a_{11} = \frac{1}{2}$, $b_{11} = \frac{F(t)}{2}$, $a_{40} + b_{40} = \frac{1}{2}$ and $b_{02} + c_{20} = \frac{3}{2}$. Hence, we constructs bilinear form for equation (4.22) as

$$(D_xD_t + F(t)D_xD_y + 3D_y^2 + D_x^4)\Phi.\Phi = 0. \tag{4.24}$$

4.3. (3+1)-dimensional equations

4.3.1. Generalized BKP equation

We consider the BKP equation[46] as

$$u_{yt} + 3u_{xz} - 3u_xu_{xy} - 3u_{2x}u_y - u_{(3x)y} = 0. \tag{4.25}$$

Step 1: Finds dispersion $w_i = \frac{-3\alpha_i\gamma_i+\alpha_i^3\beta_i}{\beta_i}$ with $u = e^{\alpha_i x+\beta_i y+\gamma_i z+w_i t}$.
Step 2: Finds $R = 2$ for the transformation $u = R(ln\Phi)_x$.
Step 3: Finds quadratic equation in Φ as

$$-\Phi_t\Phi_y + \Phi\Phi_{yt} - 3\Phi_x\Phi_z + 3\Phi\Phi_{xz} - 3\Phi_{xy}\Phi_{2x} + 3\Phi_x\Phi_{(2x)y} + \Phi_y\Phi_{3x} - \Phi\Phi_{(3x)y} = 0. \tag{4.26}$$

Step 4: Finds nonzero coefficients $b_{31} = -\frac{1}{2}$, $c_{11} = \frac{3}{2}$ and $d_{11} = \frac{1}{2}$. Hence, we constructs bilinear form for equation (4.25) as

$$(D_yD_t + 3D_xD_z - D_x^3D_y)\Phi.\Phi = 0. \tag{4.27}$$

4.3.2. Boiti-Leon-Manna-Pempinelli (BLMP) equation

Taking the BLMP equation[47] as

$$u_{yt} + u_{zt} + u_{(3x)y} + u_{3xz} - 3u_x u_{xy} - 3u_x u_{xz} - 3u_{2x} u_y - 3u_{2x} u_z = 0. \tag{4.28}$$

Step 1: Finds dispersion $w_i = -\alpha_i^3$ with $u = e^{\alpha_i x + \beta_i y + \gamma_i z + w_i t}$.
Step 2: Finds $R = -2$ for the transformation $u = R(ln\Phi)_x$.
Step 3: Finds quadratic equation in Φ as

$$-\Phi_t \Phi_y - \Phi\Phi_{zt} + \Phi_t \Phi_y - \Phi\Phi_{yt} - 3\Phi_{xz}\Phi_{2x} - 3\Phi_{xy}\Phi_{2x} + 3\Phi_x \Phi_{(2x)z} + 3\Phi_x \Phi_{(2x)y} + \Phi_z \Phi_{3x} + \Phi_y \Phi_{3x} - \Phi\Phi_{(3x)z} - \Phi\Phi_{(3x)y} = 0. \tag{4.29}$$

Step 4: Finds nonzero coefficients $b_{31} = -\frac{1}{2}$, $c_{31} = -\frac{1}{2}$, $d_{11} = -\frac{1}{2}$ and $h_{11} = -\frac{1}{2}$. Hence, we constructs bilinear form for equation (4.28) as

$$(D_t + D_x^3)(D_y + D_z)\Phi.\Phi = 0. \tag{4.30}$$

4.4. (4+1)-dimensional equation

4.4.1. (4+1)-dimensional Fokas equation

Considering the Fokas equation[48] as

$$4u_{xt} - u_{4x} + u_{x3y} + 6(u^2)_{xy} - 6u_{zs} = 0. \tag{4.31}$$

Step 1: Finds dispersion $w_i = \frac{\alpha_i^4 + 6\gamma_i\delta_i - \alpha_i\beta_i^3}{4\alpha_i}$ with $u = e^{\alpha_i x + \beta_i y + \gamma_i z + \delta_i s + w_i t}$.
Step 2: Finds $R = \frac{-\alpha_1^3 + \beta_1^3}{\alpha_1^2 \beta_1}$ for logarithmic transformation $u = R(ln\Phi)_{xx}$.
Step 3: Finds quartic equation in Φ (not showing due to lengthy expression).
Step 4: Finds nonzero coefficients and relations $a_{40} + b_{40} + c_{40} + d_{40} = -A$, $a_{11} = 4A$, $b_{13} = A$ and $p_{11} = -6A$, where $A = \frac{\alpha_1^2\beta_1^4 - \alpha_1^5\beta_1}{2}$. Hence, we constructs bilinear form for equation (4.31) as

$$A(4D_x D_t - D_x^4 + D_x D_y^3 - 6D_z D_s)\Phi.\Phi = 0, \tag{4.32}$$

for $A \neq 0$

$$(4D_x D_t - D_x^4 + D_x D_y^3 - 6D_z D_s)\Phi.\Phi = 0. \tag{4.33}$$

5. Conclusions

In conclusion, we have constructed a new and efficient algorithm to establish Hirota's bilinear form for a class of (n+1)-dimensional nonlinear PDEs. Furthermore, several examples of well-known nonlinear evolution equations, for example, the KdV equation, KP equation, SK equation, shallow water wave, BLMP equation, HSI system, generalized BKP equation, Fokas equation, and others, are calculated with the help of a proposed new algorithm using system software Mathematica. Our established results also revealed that the derived algorithm is a successful and robust tool for obtaining bilinear forms for a class of nonlinear PDEs that come from different areas of nonlinear dynamics, oceanography, mathematical physics, fluid dynamics, soliton theory, and other nonlinear sciences. Such results are extremely recommended in advanced research and innovation.

Conflict of Interest

The authors declare that they have no conflict of interest.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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