Bilinearization and new center-controlled N-rogue solutions to a (3+1)-dimensional generalized KdV-type equation in plasmas via direct symbolic approach

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Abstract: This research studies rogue wave solutions with center parameters of a generalized (3+1)-dimensional KdV-type equation in plasma physics. Using a direct generalized formula and symbolic technique, it creates rogue waves with adjustable dynamical characteristics controlled by the center parameters. Our investigation produces rogue wave solutions up to the third order with Painlevé transformation through direct computation, considering a range of center-controlled and other parameters within the investigated equation. To facilitate our analysis, we derive an equation for the function f in bilinear form, utilize the Cole-Hopf transformation for the wave function u, and introduce the transformed variable ζ. It obtains the rogue wave solutions using a generalized form for N-rogue waves generated from Hirota's N-soliton approach. Through the powerful symbolic computation tool *Mathematica*, we provide visualizations of the dynamic behavior of rogue waves across diverse center parameters. This research highlights the prevalence of massive rogue waves within nonlinear phenomena, showcasing their dominance over their smaller counterparts. The investigated equation offers insights into the evolution of longer waves characterized by smaller amplitudes, which is particularly relevant in plasma physics, fluid dynamics, and dispersive media. Rogue waves have applications in diverse scientific fields, including oceanography, nonlinear dynamics, fluid dynamics, fiber optics, dusty plasma, and complex nonlinear systems.

Keywords: Bilinear form; Cole-Hopf transformation; Dispersion; N-rogue waves; Higher-order rogues.

1 Introduction

A large area of physics and applied mathematics deals with nonlinear partial differential equations (PDEs) [1–8], which involve unknown functions and its partial derivatives. PDEs model the complex physical processes in several engineering and nonlinear sciences. Nonlinear PDEs have been used by mathematicians to explain a variety of physical phenomena, from fluid dynamics to gravitational studies, and to provide solutions to problems such as the Poincaré and Calabi conjectures. Since there are no general techniques for solving nonlinear PDEs, analyzing them and finding solutions can be difficult. PDEs [9–17] are widely used in diverse nonlinear sciences to model and understand physical phenomena that involve multiple variables and their rates of change. The different examples of PDEs include the heat equation [18], the wave equation [19], and the famous Schrödinger equation [20] from quantum mechanics. Numerous techniques have been used to address nonlinear PDEs to find exact and analytical solutions such as the Bäcklund transformation [1,2], Hirota's bilinearization method [3,4], inverse scattering method [5,6], the simplified Hirota's approach [7,8],

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the Darboux transformation [9, 10], the Lie symmetry analysis [11, 12], bilinear neural network method [13–15], the Pfaffian technique [16, 17], and other techniques.
Rogue waves [21–24] are space-time localized waves with a large amplitude, sometimes called giant waves that arise from nowhere in the ocean. They could seriously hurt people since they are unpredictable and dangerous. It is essential to investigate how rogue waves evolve, which interests many academics. Rogue waves are characterized by their height, sometimes even more than nearby waves. Because rogue waves defy accepted linear wave models, they are studied in nonlinear wave dynamics. Scientific research on extreme or giant waves aims to forecast their occurrence and understand their underlying mechanics. Rogue waves suddenly emerge when shorter waves focus on a small area with their energy. One noteworthy application is the enhancement of marine safety. The developing models and prediction algorithms provide early detection and warnings to prevent harm caused by these waves. This information may be helpful in offshore gas and oil fields, the maritime industry, and the coastal area. Thus, understanding rogue waves' dynamics helps design secure structures and develop strategies to lower their impacts. So, achieving excellent operational safety and more affordable solutions is possible. Additionally, investigating their origins and dynamics extends our knowledge of complex procedures and the emergence of severe events in diverse mathematical and physical contexts.
This work examines newly constructed center-controlled rogue waves with Painlevé tranformation to an integrable (3+1)-D generalized KdV-type equation [25] in plasma physics as

$$u_{xxxy} + \lambda_1 u_{xt} + \lambda_2 u_{yt} + \lambda_3 (u_x u_y)_x + \lambda_4 u_{xx} + \lambda_5 u_{zz} = 0, \tag{1}$$

where $u(x, y, z, t)$ is a wave function of spatial coordinates x, y, z and temporal coordinate t, and $\lambda_k; 1 \leq k \leq 5$ are real parameters. The equation consists of mainly dispersive, nonlinear, and disturbed terms which generalizes the well-known equations for different parameter values as

- (3+1)-D Hirota bilinear equation [26] for $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = \lambda_4 = 3, \lambda_5 = -3$ as

$$u_{xxxy} + 3(u_x u_y)_x + 3u_{xx} - 3u_{zz} - u_{yt} = 0. \tag{2}$$

- (3+1)-D KP equation [27] for $\lambda_1 = \lambda_2 = 1, \lambda_3 = 3, \lambda_4 = 0, \lambda_5 = -1$ as

$$u_{xxxy} + 3(u_x u_y)_x - u_{zz} + u_{xt} + u_{yt} = 0. \tag{3}$$

The Korteweg-de Vries (KdV) type equations refer to a class of nonlinear PDEs such as modified KdV equation [28], generalized KdV equation [29], Boussinesq equation [30] and others that share similar mathematical structures with the KdV equation. These equations often describe wave phenomena and can be derived from various physical systems. They may include additional terms or modifications to the original KdV equation to represent different physical phenomena or to incorporate additional effects. The KdV-type equations and their variants play a significant role in plasma physics, particularly in describing the behavior of nonlinear waves and structures in plasmas. In plasma physics, the KdV-type equations study solitary waves, nonlinear wave interactions, plasma turbulence, collisionless shocks, and plasma transport. Recently, in 2023, Kumar-Mohan [25] proposed this equation (1) and studied the Painlevé integrability, and discussed the rogue waves with center-parameter by proposing a direct symbolic approach. The present work constructs the bilinear form for the studied equation. It analyzes the new center-controlled rogue waves by transforming the equation with Painlevé transformation and symbolically utilizes the proposed direct approach. The Painlevé transformation transformed the examined equation to a new (1+1)-dimensional evolution equation in new transformed variables, which can be converted to Hirota's bilinear form in auxiliary function with Cole-Hopf transformation. Due to the Painlevé transformation to this studied equation, we observe the new rogue waves and study their dynamic behavior. Since this equation is newly proposed, no more work on this

equation is found in the literature. However, due to the applicability of this KdV-type integrable equation, several types of solutions, such as bright and dark solitons, lumps, kinks, and breathers, can be studied.
An integrable nonlinear PDE (NLPDE) has solutions that are localized in nature within particular directions, such as lumps [31], optical solitons [32, 33], solitons [34, 35], and other solutions. Examining the integrability of NLPDEs can help to find exact and closed-form solutions. For a NLPDE, the Painlevé test can confirm integrability completely [36, 37]. It is very laborious to determine the integrability of a NLPDE with the Painlevé analysis, but computer algebra systems such as *Mathematica*, *Maple*, and other tools allow one to perform this analysis. In order to precisely understand the properties of different facts in different natural science fields, we search for specific answers. As earlier mentioned, nonlinear PDE has attracted the interest of numerous researchers due to its resemblance to real-world problems and its capacity to yield several solutions. The system software can help locate different kinds of answers about integrability, lax pairs, and the existence of several solutions, such as solitons, lumps, breathers, rogues, and dynamical behavior. An exciting area of research to demonstrate basic concepts in shallow water waves, engineering sciences, oceanography, dusty plasma, and other nonlinear systems has been the dynamic behavior of rogue waves developed for NLPDEs.

The following section describes the direct symbolic approach for finding the rogue wave solutions of the studying equation. Section 3 transforms the said equation using Painlevé transformation and finds the dispersion and bilinear form for the transformed equation by applying the Cole-Hopf transformation. In Section 4, we obtain 1^{st}-, 2^{nd}-, and 3^{rd}-order rogue waves and showcase the dynamical behavior for these solutions with reasonable values of center parameters. Section 5 discusses the results with the findings, and the last section concludes the research study.

2 Direct symbolic approach for rogue wave solutions

Let us assume a nonlinear evolution equation of (3+1)-dimensions as

$$S(u, u_x, u_y, u_z, u_t, u_{xx}, u_{xt}, u_{xy}, u_{xxx}, \cdots) = 0, \tag{4}$$

which contains partial derivatives with independent variables $\{x, y, z, t\}$ to unknown function u.
Using Painlevé transformation

$$u = u(\zeta, z), \qquad \zeta(x, y, t) = c_1 x + c_2 y - c_3 t \tag{5}$$

where $c_i; 1 \leq i \leq 3$ are non-zero constants, the equation (4) is transformed to an (1+1)-dimensional PDE as

$$T(u, u_\zeta, u_z, u_{\zeta z}, u_{\zeta\zeta}, u_{zz}, \cdots) = 0. \tag{6}$$

Now, constructing the Cole-Hopf transformation

$$u(\zeta, z) = K(\ln f)_{\zeta^s}, \tag{7}$$

where K is a nonzero real constant and $f(\zeta, z)$ is an auxiliary function in ζ and z, where s is the order of ζ obtained by balancing the nonlinear and higher-order terms in PDE (6). This transformation transforms the equation (6) to a bilinear equation in auxiliary function f, which can be written in bilinear form with D-operators.
We find the rogue wave solutions by considering the auxiliary function $f(\zeta, z)$ as a generalized form [25, 38]

$$f(\zeta, z) = \widehat{F_n}(\zeta, z, \alpha, \beta) = \sum_{s=0}^{\frac{n(n+1)}{2}} \sum_{i=0}^{s} q_{n(n+1)-2s,2i} (\zeta - \alpha)^{n(n+1)-2s} (z - \beta)^{2i}, \tag{8}$$

where $q_{l,m}; l, m \in \{0, 2, \cdots, s(s+1)\}$ and $\{\alpha, \beta\}$ are constants and center parameters, respectively.

3 Logarithmic transformation and bilinear form

The Painlevé transformation $u = u(\zeta, z)$ with $\zeta = x+y-t$ in equation (1), generates a transformed equation as

$$u_{\zeta\zeta\zeta\zeta} - (\lambda_1 + \lambda_2 - \lambda_4)u_{\zeta\zeta} + 2\lambda_3 u_\zeta u_{\zeta\zeta} - \lambda_5 u_{\mathrm{zz}} = 0. \tag{9}$$

Considering phase θ_i for the equation (9)

$$\theta_i = p_i\zeta - w_i z, \tag{10}$$

with p_i and w_i for $i \in N$ as the constants and dispersions, respectively. By collecting linear terms in Eq. (9) and putting $u(\zeta, z) = e^{\theta_i}$, we find

$$w_i = \frac{\pm p_i\sqrt{p_i^2 - (\lambda_1 + \lambda_2 - \lambda_4)}}{\sqrt{\lambda_5}}. \tag{11}$$

Now, considering Cole-Hopf transformation

$$u(\zeta, z) = R(\ln f)_\zeta, \tag{12}$$

and find the constant R by substituting the equation (12) with auxiliary function as one soliton solution $f(\zeta, z) = 1 + e^{\theta_1}$ in equation (9) as

$$R = \frac{6}{\lambda_3}.$$

Therefore, the equation (12) becomes

$$u(\zeta, z) = \frac{6}{\lambda_3}(lnf)_\zeta. \tag{13}$$

On substituting the transformation (13) into transformed equation (9), gives a bilinear equation in $f = f(\zeta, z)$ as

$$\left(3f_{\zeta\zeta}^2 - 4f_\zeta f_{\zeta\zeta\zeta} + ff_{\zeta\zeta\zeta\zeta}\right) - (\lambda_1 + \lambda_2 - \lambda_4)\left(ff_{\zeta\zeta} - f_\zeta^2\right) - \lambda_5\left(ff_{\mathrm{zz}} - f_{\mathrm{z}}^2\right) = 0, \tag{14}$$

which can be constructed in bilinear D-operator form. Hirota [3] defined differential operators $D_i : i = x, y, z, s, t$ as

$$D_x^{r_1} D_y^{r_2} U(x,y)V(x,y) = \left(\frac{\partial}{\partial_x} - \frac{\partial}{\partial_{x'}}\right)^{r_1}\left(\frac{\partial}{\partial_y} - \frac{\partial}{\partial_{y'}}\right)^{r_2} U(x,y)V(x',y')|_{x=x',y=y'},$$

where x', y' and and $r_i : i = 1, 2$ are the formal variables and positive integers, respectively. We get the required definitions of D-operators as

$$\begin{aligned} D_\xi^2 f.f &= 2(ff_{\xi\xi} - f_\xi^2); \quad \xi : \zeta, z \\ D_\zeta^4 f.f &= 2(3f_{\zeta\zeta}^2 - 4f_\zeta f_{\zeta\zeta\zeta} + ff_{\zeta\zeta\zeta\zeta}). \end{aligned} \tag{15}$$

Thus, the equation (14) can be written in the Hirota's bilinear form as

$$\left[D_\zeta^4 - (\lambda_1 + \lambda_2 - \lambda_4)D_\zeta^2 - \lambda_5 D_z^2\right] f.f = 0. \tag{16}$$

4 Solutions for rogue waves with center parameters

4.1 1^{st}-order rogue waves

For $n = 1$ in equation (8), we consider the function f as

$$f(\zeta, z) = q_{0,0} + q_{0,2}z^2 + q_{2,0}\zeta^2. \tag{17}$$

By putting the equation (17) into (14), and equating all coefficients of different powers of ζ and z, we get the system of equations as

$$\begin{aligned} 2q_{2,0}\left((\lambda_1+\lambda_2-\lambda_4)\,q_{2,0}-\lambda_5 q_{0,2}\right) &= 0, \\ 2q_{0,2}\left(\lambda_5 q_{0,2}-(\lambda_1+\lambda_2-\lambda_4)\,q_{2,0}\right) &= 0, \\ (\lambda_1+\lambda_2-\lambda_4)\,q_{0,0}q_{2,0}+\lambda_5 q_{0,0}q_{0,2}-6q_{2,0}^2 &= 0, \end{aligned} \tag{18}$$

which give values for the constants by solving these equations as

$$q_{0,0} = \frac{3q_{2,0}}{\lambda_1+\lambda_2-\lambda_4}, \qquad q_{0,2} = \frac{(\lambda_1+\lambda_2-\lambda_4)\,q_{2,0}}{\lambda_5}, \qquad q_{2,0} = q_{2,0}. \tag{19}$$

So, the equation (17) becomes

$$f(\zeta, z) = \widehat{F_1}(\zeta, z, \alpha, \beta) = \left[(\zeta-\alpha)^2 + \frac{3}{\lambda_1+\lambda_2-\lambda_4} + \frac{(\lambda_1+\lambda_2-\lambda_4)\,(z-\beta)^2}{\lambda_5}\right] q_{2,0}, \tag{20}$$

and gives a solution of Eq. (16), having α and β as center parameters. The solution is obtained by putting the equation (20) into (13) as

$$u(\zeta, z) = \widehat{u_1}(\zeta, z, \alpha, \beta) = \frac{12(\zeta-\alpha)}{\lambda_3\left((\alpha-\zeta)^2 + \frac{3}{\lambda_1+\lambda_2-\lambda_4} + \frac{(\lambda_1+\lambda_2-\lambda_4)(z-\beta)^2}{\lambda_5}\right)}, \tag{21}$$

4.2 2^{nd}-order rogue waves

For $n = 2$ in equation (8), we take the function f as

$$f(\zeta, z) = q_{0,0} + q_{0,2}z^2 + q_{0,4}z^4 + q_{0,6}z^6 + q_{2,0}\zeta^2 + q_{2,2}\zeta^2z^2 + q_{2,4}\zeta^2z^4 + q_{4,0}\zeta^4 + q_{4,2}\zeta^4z^2 + q_{6,0}\zeta^6. \tag{22}$$

On substituting the equation (22) into (16), we get a system by equating the coefficients for powers of ζ and z to zero. Form this system, we determine the values as

$$\begin{gathered} q_{0,0} = \frac{625\lambda_5 q_{4,2}}{\lambda^4}, \qquad q_{0,2} = \frac{475 q_{4,2}}{3\lambda^2}, \qquad q_{0,4} = \frac{17q_{4,2}}{3\lambda_5}, \\ q_{0,6} = \frac{\lambda^2 q_{4,2}}{3\lambda_5^2}, \qquad q_{2,0} = -\frac{125\lambda_5 q_{4,2}}{3\lambda^3}, \qquad q_{2,2} = \frac{30q_{4,2}}{\lambda}, \\ q_{2,4} = \frac{\lambda q_{4,2}}{\lambda_5}, \qquad q_{4,0} = \frac{25\lambda_5 q_{4,2}}{3\lambda^2}, \qquad q_{4,2} = q_{4,2} \\ q_{6,0} = \frac{\lambda_5 q_{4,2}}{3\lambda}, \end{gathered} \tag{23}$$

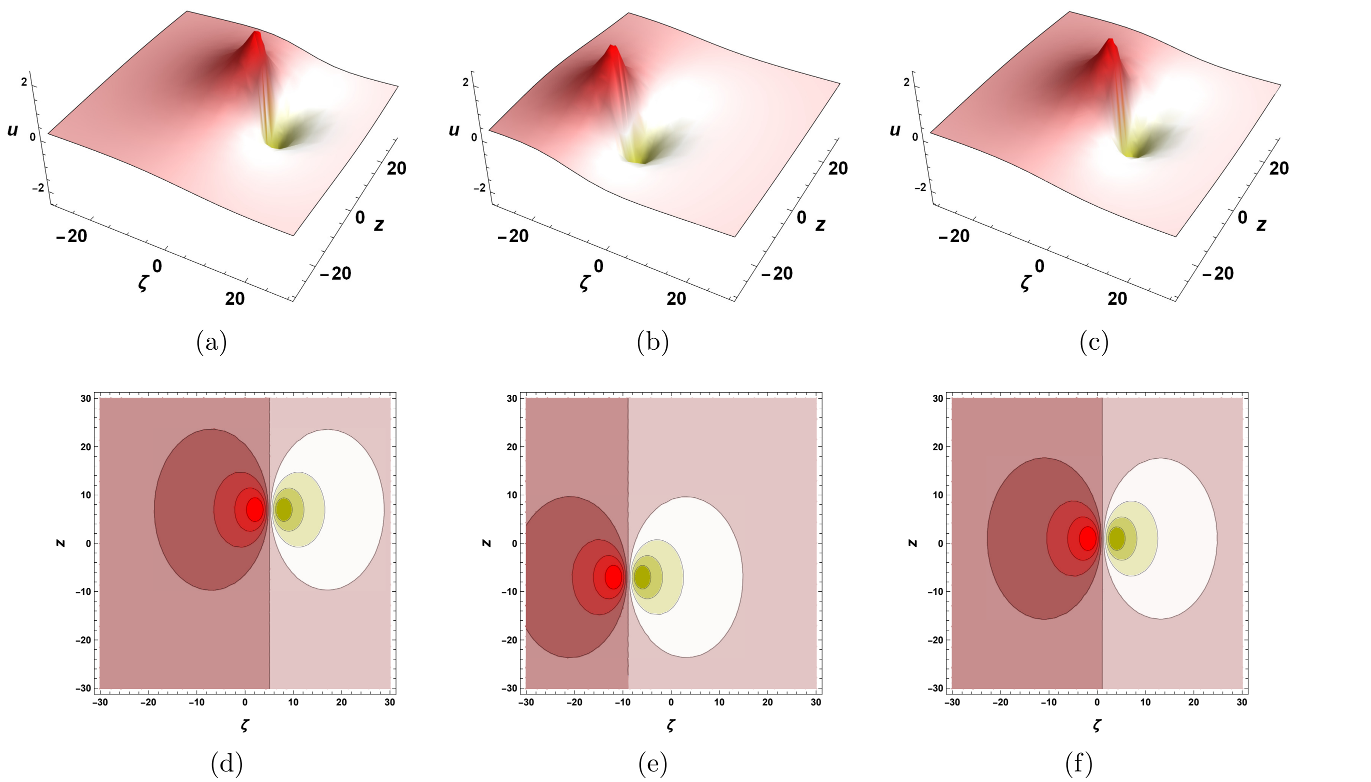

Figure 1: 1^{st}-order rogue waves for (21) with values: $\lambda_1 = 0.5, \lambda_2 = 1, \lambda_3 = -1, \lambda_4 = \lambda_5 = 1$, and parameters as: **(a)** $\alpha = 5, \beta = 7$; **(b)** $\alpha = -9, \beta = -7$; and **(c)** $\alpha = 1, \beta = 1$. **(d)-(f)** are contour plots of **(a)-(c)** in ζz-plane.

where $\lambda = (\lambda_1 + \lambda_2 - \lambda_4)$. Thus, the equation (17) becomes

$$f(\zeta, z) = \widehat{F_2}(\zeta, z, \alpha, \beta) = \frac{q_{4,2}}{3}\Big(\frac{\lambda_5(\alpha-\zeta)^6}{\lambda_1+\lambda_2-\lambda_4} + \frac{25\lambda_5(\alpha-\zeta)^4}{(\lambda_1+\lambda_2-\lambda_4)^2} - \frac{125\lambda_5(\alpha-\zeta)^2}{(\lambda_1+\lambda_2-\lambda_4)^3} + \frac{1875\lambda_5}{(\lambda_1+\lambda_2-\lambda_4)^4} + \frac{3(\lambda_1+\lambda_2-\lambda_4)(\alpha-\zeta)^2(z-\beta)^4}{\lambda_5} + \frac{90(\alpha-\zeta)^2(z-\beta)^2}{\lambda_1+\lambda_2-\lambda_4} + 3(\alpha-\zeta)^4(z-\beta)^2 + \frac{(\lambda_1+\lambda_2-\lambda_4)^2(z-\beta)^6}{\lambda_5^2} + \frac{17(z-\beta)^4}{\lambda_5} + \frac{475(z-\beta)^2}{(\lambda_1+\lambda_2-\lambda_4)^2}\Big), \quad (24)$$

and gives a solution of Eq. (14) having α and β as center parameters. We get the solution by putting the equation (24) into (13) as

$$u(\zeta, z) = \frac{6}{\lambda_3}(\ln \widehat{F_2}(\zeta, z, \alpha, \beta))_{\zeta}. \quad (25)$$

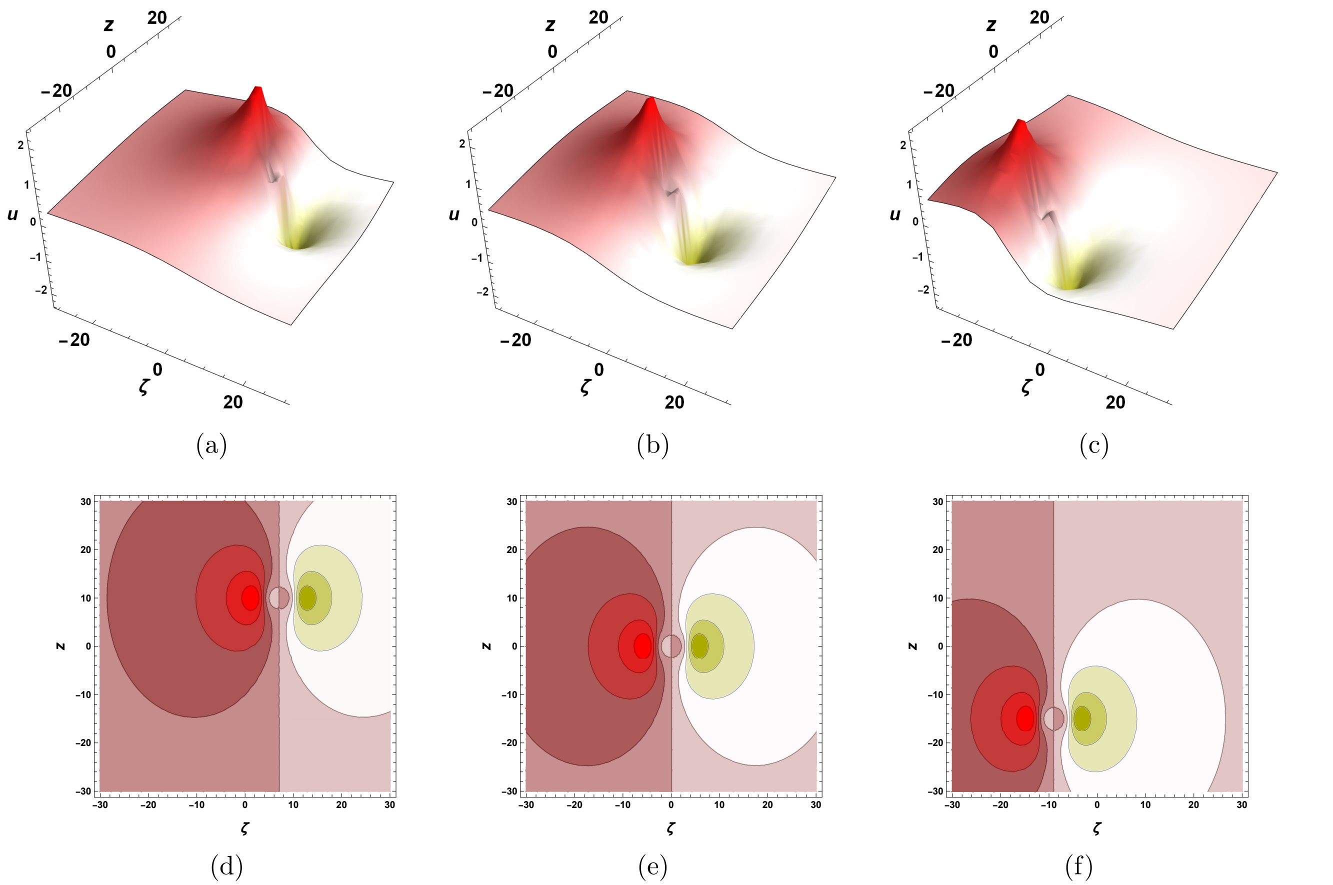

Figure 2: 2^{nd}-order rogue waves for (25) with values: $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = -2, \lambda_4 = \lambda_5 = 1$, and parameters as: **(a)** $\alpha = 7, \beta = 10$; **(b)** $\alpha = 0, \beta = 0$; and **(c)** $\alpha = -9, \beta = -15$. **(d)-(f)** are contours of **(a)-(c)** in ζz-plane.

4.3 3^{rd}-order rogue waves

For $n = 3$ in equation (8), we get rogue waves of third-order for the function f as

$$\begin{aligned} f(\zeta, z) = {}& q_{0,0} + q_{0,2}z^2 + q_{0,4}z^4 + q_{0,6}z^6 + q_{0,8}z^8 + q_{0,10}z^{10} + q_{0,12}z^{12} + q_{2,0}\zeta^2 + q_{2,2}z^2\zeta^2 + q_{2,4}\zeta^2 z^4 \\ & + q_{2,6}\zeta^2 z^6 + q_{2,8}\zeta^2 z^8 + q_{2,10}\zeta^2 z^{10} + q_{4,0}\zeta^4 + q_{4,2}\zeta^4 z^2 + q_{4,4}z^4\zeta^4 + q_{4,6}\zeta^4 z^6 + q_{4,8}\zeta^4 z^8 + q_{6,0}\zeta^6 \\ & + q_{6,2}\zeta^6 z^2 + q_{6,4}\zeta^6 z^4 + q_{6,6}\zeta^6 z^6 + q_{8,0}\zeta^8 + a_{8,2}\zeta^8 z^2 + q_{8,4}\zeta^8 z^4 + q_{10,0}\zeta^{10} + q_{10,2}\zeta^{10} z^2 + q_{12,0}\zeta^{12}. \end{aligned} \quad (26)$$

On having Eq. (22) into Eq. (16), we obtain a system by equating all the coefficients for powers of ζ and z to zero. By solving it, we determine the values as

$$q_{0,0} = \frac{878826025\lambda_5 q_{10,2}}{54\lambda^7}, \qquad q_{0,2} = \frac{150448375 q_{10,2}}{9\lambda^5}, \qquad q_{0,4} = \frac{16391725 q_{10,2}}{18\lambda^3\lambda_5},$$

$$q_{0,6} = \frac{399490 q_{10,2}}{9\lambda\lambda_5^2}, \qquad q_{0,8} = \frac{1445\lambda q_{10,2}}{2\lambda_5^3}, \qquad q_{0,10} = \frac{29\lambda^3 q_{10,2}}{3\lambda_5^4},$$

$$q_{0,12} = \frac{\lambda^5 q_{10,2}}{6\lambda_5^5}, \qquad q_{2,0} = \frac{79893275\lambda_5 q_{10,2}}{9\lambda^6}, \qquad q_{2,2} = \frac{94325 q_{10,2}}{\lambda^4}, \qquad q_{2,4} = -\frac{2450 q_{10,2}}{\lambda^2\lambda_5},$$

$$q_{2,6}=\frac{17710q_{10,2}}{3\lambda_5^2},\quad q_{2,8}=\frac{95\lambda^2 q_{10,2}}{\lambda_5^3},\quad q_{2,10}=\frac{\lambda^4 q_{10,2}}{\lambda_5^4},$$

$$q_{4,0}=-\frac{5187875\lambda_5 q_{10,2}}{18\lambda^5},\quad q_{4,2}=\frac{36750q_{10,2}}{\lambda^3},\quad q_{4,4}=\frac{18725q_{10,2}}{3\lambda\lambda_5},$$

$$q_{4,6}=\frac{730\lambda q_{10,2}}{3\lambda_5^2},\quad q_{4,8}=\frac{5\lambda^3 q_{10,2}}{2\lambda_5^3},\quad q_{6,0}=\frac{37730\lambda_5 q_{10,2}}{9\lambda^4},$$

$$q_{6,2}=\frac{9310q_{10,2}}{3\lambda^2},\quad q_{6,4}=\frac{770q_{10,2}}{3\lambda_5},\quad q_{6,6}=\frac{10\lambda^2 q_{10,2}}{3\lambda_5^2},$$

$$q_{8,0}=\frac{245\lambda_5 q_{10,2}}{2\lambda^3},\quad q_{8,2}=\frac{115q_{10,2}}{\lambda},\quad q_{8,4}=\frac{5\lambda q_{10,2}}{2\lambda_5},\quad q_{10,0}=\frac{49\lambda_5 q_{10,2}}{3\lambda^2}$$

$$q_{10,2}=q_{10,2},\quad q_{12,0}=\frac{\lambda_5 q_{10,2}}{6\lambda}, \tag{27}$$

where $\lambda=(\lambda_1+\lambda_2-\lambda_4)$. So, the equation (17) becomes

$$\begin{aligned}f(\zeta,z)=\widehat{F_3}(\zeta,z,\alpha,\beta)=\frac{q_{10,2}}{54}\Big(&\frac{180A^6B^6C^2}{\lambda_5^2}+\frac{135A^4B^8C^3}{\lambda_5^3}-\frac{132300A^2B^4}{C^2\lambda_5}+\frac{5130A^2B^8C^2}{\lambda_5^3}+\frac{54A^2B^{10}C^4}{\lambda_5^4}\\
&+\frac{167580A^6B^2}{C^2}+\frac{1984500A^4B^2}{C^3}+\frac{5093550A^2B^2}{C^4}+\frac{135A^8B^4C}{\lambda_5}+\frac{337050A^4B^4}{C\lambda_5}+\frac{13140A^4B^6C}{\lambda_5^2}\\
&+\frac{6210A^8B^2}{C}+\frac{13860A^6B^4}{\lambda_5}+\frac{318780A^2B^6}{\lambda_5^2}+54A^{10}B^2+\frac{882A^{10}\lambda_5}{C^2}+\frac{6615A^8\lambda_5}{C^3}+\frac{226380A^6\lambda_5}{C^4}\\
&-\frac{15563625A^4\lambda_5}{C^5}+\frac{479359650A^2\lambda_5}{C^6}+\frac{9A^{12}\lambda_5}{C}+\frac{49175175B^4}{C^3\lambda_5}+\frac{522B^{10}C^3}{\lambda_5^4}+\frac{9B^{12}C^5}{\lambda_5^5}+\frac{902690250B^2}{C^5}\\
&+\frac{2396940B^6}{C\lambda_5^2}+\frac{39015B^8C}{\lambda_5^3}+\frac{878826025\lambda_5}{C^7}\Big),\end{aligned} \tag{28}$$

where $A=(\alpha-\zeta), B=(z-\beta)$ and $C=(\lambda_1+\lambda_2-\lambda_4)$. We get the solution by putting the equation (28) into (13) as

$$u(\zeta,z)=\frac{6}{\lambda_3}(\ln\widehat{F_3}(\zeta,z,\alpha,\beta))_\zeta. \tag{29}$$

5 Results and findings

The studied equation (1) is completely integrable, so this can show a wide range of solutions, such as solitary waves, kinks, rogues, breathers, and lumps. Using the discussed symbolic computation approach, we found the 1^{st}-, 2^{nd}-, and 3^{rd}-order center-controlled rogue waves. Two and three rogue wave solutions illustrate the interaction solutions with center-controlled parameters, showing the dominant behavior of the large rogue wave over the smaller rogue waves. Interactions of the rogue waves occur below the surface levels concerning the singularities for all the graphics. The dynamic behavior of the solutions with the appropriate parameter selections can explain the following results:

- Figure 1 illustrates the 1^{st}-order rogue waves with singularity around center parameter $\zeta=\alpha$, the constants $\lambda_1=0.5,\lambda_2=1,\lambda_3=-1,\lambda_4=\lambda_5=1$, and $\{\alpha,\beta\}$ as $\alpha=5,\beta=7$; $\alpha=-9,\beta=-7$; and $\alpha=1,\beta=1$, for (a)-(c). A single rouge wave in all the graphics shows quasi-bright-dark solitary wave behavior with respect to the singularity. When the center-controlled parameter changes, the rogue waves change their positions to the considered center values.

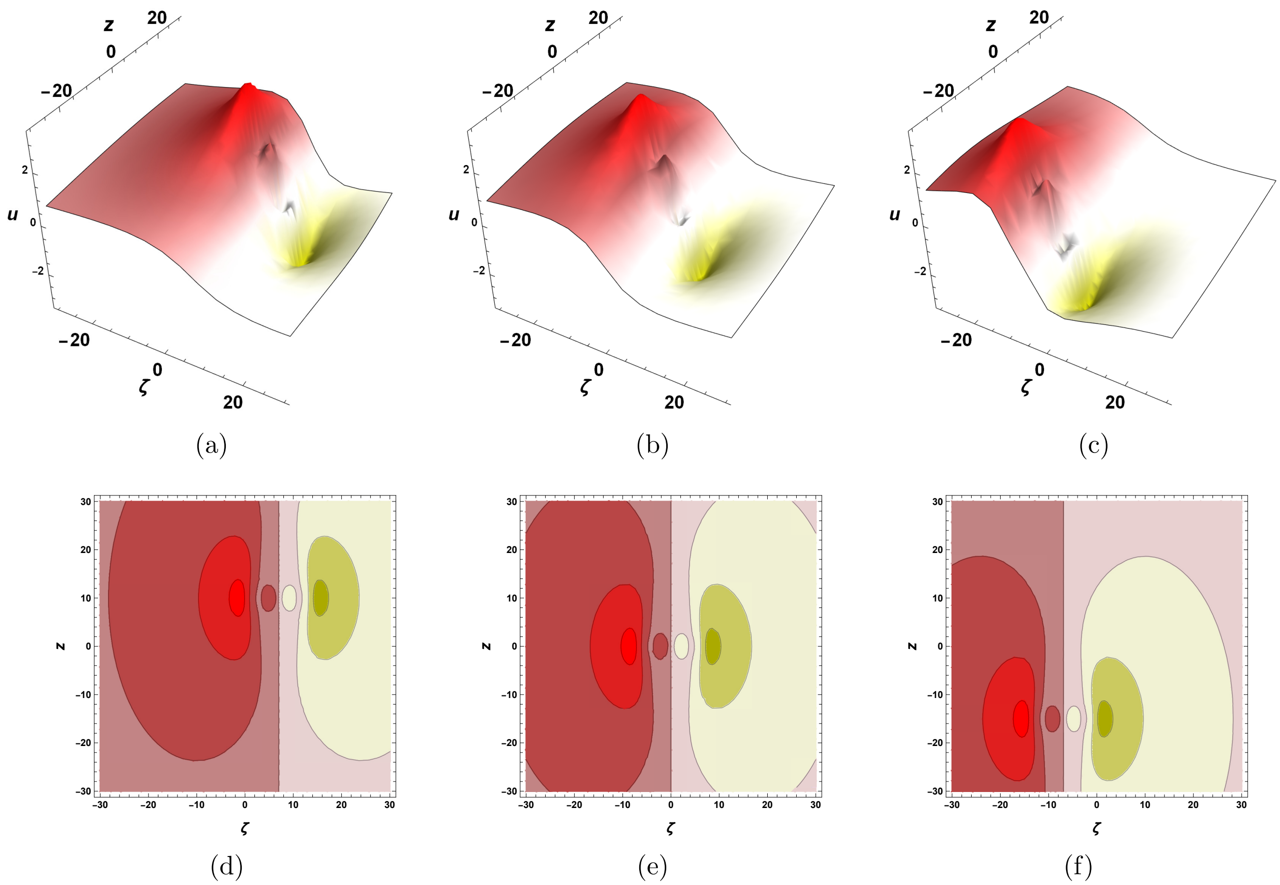

Figure 3: 3^{rd}-order rogue wave solutions for (29) with values: $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = -2, \lambda_4 = 1, \lambda_5 = 2$, and parameters as: **(a)** $\alpha = 7, \beta = 10$; **(b)** $\alpha = 0, \beta = 0$; and **(c)** $\alpha = -7, \beta = -15$. **(d)-(f)** are contours for **(a)-(c)** in ζz-plane.

- In figure 2, we show the 2^{nd}-order rogues along with center parameters (α, β). It displays the formation of two rogue waves that are formed with respect to the singularities and the large rogue wave have the dominating nature to other rogue, with the constants $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = -2, \lambda_4 = \lambda_5 = 1$, and the parameters as $\alpha = 7, \beta = 10$; $\alpha = 0, \beta = 0$; and $\alpha = -9, \beta = -15$, for (a)-(c). All the graphics show the interaction of two rogue waves, which are shown in contour plots with small circle-like shapes at the intersection line. Rogue waves change their positions relative to the chosen center-controlled parameters.

- Figure 3 showcaes the dynamics for 3^{rd}-order rogues along with parameters (α, β). It shows the formation of three rogue waves that are formed with respect to the singularities and the large rogue waves have the dominating nature to smaller rogue waves, with the constants $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = -2, \lambda_4 = 1, \lambda_5 = 2$ and the parameters as $\alpha = 7, \beta = 10$; $\alpha = 0, \beta = 0$; and $\alpha = -7, \beta = -15$ for (a)-(c). The graphics in this figure display the interaction behavior of three rogue waves. In contour plots, the two interactions of these three rogue waves are shown as two elliptic shapes near the interaction line. These interactions show the dominant behavior of more giant rogue waves over smaller rogue waves, which change positions along with center-controlled parameters.

6 Conclusions

In this work, an integrable (3+1)-dimensional generalized KdV-type equation with Painlevé transformation was studied. With center-controlled parameters α and β, we generated the rogue waves using a symbolic computation approach. The interaction behaviors of the rogue waves were shown for the two and three rogue waves; these interactions showed the dominant behavior of more giant rogue waves over smaller rogue waves, which change positions along with center-controlled parameters. We obtained 1^{st}-, 2^{nd}-, and 3^{rd}-order rogue wave solutions with appropriate selections of different constants in the governed equation and distinct values of center-controlled parameters. The bilinear form and the Cole-Holf transformation of the transformed equation were constructed. We used a generalized N-rogue wave expression by Hirota's N-soliton solution approach to establish the rogue waves. With the computer algebra system *Mathematica*, we illustrated the dynamic behavior of the obtained solutions with appropriate values for the center parameter. Our findings showed that the singularities in rogue waves occur along with the parameters α and β that control the center. The studied equation examines the propagation of longer waves with smaller amplitudes in plasmas, the motion of waves in fluid dynamics, and weakly dispersive waves in other mediums.
The studied (3+1)-D generalized NLPDE generalizes the KP and Hirota bilinear equations. Researchers can study wave solutions for this generalized equation, including bright and dark solitons, lumps, kinks, and breathers. Both known equations generalized by the examined equation have occurrences in plasmas, oceanography, and several other fields. Since we obtained the rogue wave solutions by applying the symbolic computational approach, there are considerable opportunities to investigate this generalized equation by employing other approaches and techniques.

Declarations

Ethics approval and consent to participate

Not applicable.

Competing interests

The authors state that there is no conflict of interest.

Authors' contributions

Both authors have agreed and given their consent for the publication of this research paper.

Data availability statement

No data from other sources has been used in this study.

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