DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DELHI, DELHI – 110007
Fixation of Pre-Ph.D. Seminar

Date: 06-08-2025

Research Scholar's details	
Name: **BRIJ MOHAN**	Enrolment Number: [redacted]
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Date of initial registration: 07–02–2022	Period of extension (if any): NA
Registration valid up to: 06–02–2028	

Supervisor's details	
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Mobile: [redacted]	University of Delhi, Delhi-110007
Co-supervisor (if any): NA	
No of Publications: In SCI/SCIE journals _5_ In Scopus but not SCI/SCIE ____ MR/ZBL _____ Total: __5__	
Proposed Title of the Thesis: *Generalization of Equations, Solutions, and Techniques for KdV-type Evolution Equations in Oceanography and Nonlinear Sciences*	
[redacted] Signature of Research Scholar	[redacted] Signature of Supervisor(s)
Title finalized at the Pre-Ph.D. seminar:	
Date:	Convener, Pre-Ph.D. Seminar Committee

Attachments:

1. Copy of joining report(s)
2. Letter of extension from BRS (if any)
3. List of publications (include names of all authors; MR/ZBL numbers, Impact factor, MCQ, SCImago if any)
4. Reprints/preprints/acceptance letter
5. Attached copy of NOC from co-author(s) other than supervisors.

(9)

Dated: 07/02/2022

The Chairperson
Board of Research Studies (Mathematical Sciences)
University of Delhi
Delhi-110007

Through: The Supervisor(s) and Head of the Department.

Subject: **Joining Report**

Dear Madam,

In compliance of the Memorandum No.BRS(MS)/248/2022/[redacted] Dated 25/01/2022 regarding my provisional admission to the Ph.D Course in the Department Mathematics having paid the necessary fees to the University through online dated 07/02/2022 (photocopy of the Receipt enclosed), I hereby submit my **Joining Report**.

I am fully aware that my admission to Ph.D Course is provisional subject to:-

(i) Equivalence of my Degrees/Certificates.
(ii) Completion of other formalities like defence of the thesis topic in a departmental seminar.
(iii) Successful completion of Course Work/Research Publication Ethics/ Research Methodology paper to be assigned by the Departmental Research Committee within initial one or two semesters.

I hereby undertake to abide by the final decision of the University with regard to confirmation cancellation of my provisional admission at any stage.

I am also submitting/producing herewith originals of the following documents/certificates as required for verification:

(i) Undergraduate and postgraduate (qualifying) Degrees/Certificates.
(ii) Mark-sheets of undergraduate and postgraduate (qualifying) examinations.
(iii) Matriculation/Higher Secondary/Secondary School Certificate for verification of date of birth.
(iv) Certificate, in the case of SC/ST/OBC/PH/EWS category.

Yours faith[redacted]

(Sign[redacted] of the candidate)

Name BRIJ MOHAN
Address: 30/5A, 2nd Floor, West Patel Nagar, New Delhi
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[redacted] 07/02/2022

Signature of Supervisor
Name: Dr. Raj Kumar
Department of Mathematics
Address: KMC, University of Delhi, University Enclave, Delhi-110007
Phone/Mobile no. [redacted]

Signature of Joint Supervisor
Name:
Department of
Address:
Phone/Mobile no.

Maths/2022 [redacted]
10/2/2022

[redacted]

(Signature of the Head with seal)
अध्यक्ष/Head
गणित विभाग
Department of Mathematics

BOARD OF RESEARCH STUDIES (MATHEMATICAL SCIENCES)
University of Delhi
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Prof. Ruchi Das
Chairperson

No.[redacted]
Dated: January 25, 2022

MEMORANDUM

With reference to his application dated 01.01.2022 **Mr. Brij Mohan** is hereby informed that the Board of Research Studies (Mathematical Sciences) at its meeting held on 24.01.2022 has approved his case for provisional admission to the Ph.D. Course under **Category-1(4) of Ordinance VI** in the **Department of Mathematics** University of Delhi, w.e.f. the date of his joining under the supervision of Dr. Raj Kumar, Kirori Mal College, University of Delhi, University Enclave, Delhi- 110007 with the following research topic:

"Nonlinear Partial Differential Equations"

He should note that his provisional admission is subject to:

I. His successful completion of Course Work to be assigned by the Departmental Research Committee within initial one or two semesters.
II. Completion of other formalities like defence of the thesis topic in a departmental seminar.

He is advised to contact his supervisor(s) for further instructions and guidance in this regard.

He is required to complete the following formalities of provisional admission within four weeks from the date of dispatch of this letter failing which his admission shall be treated as cancelled without any further notice:

(a) To submit the following certificates in original:

(i) Undergraduate and postgraduate (qualifying) Degrees/ Certificates.
(ii) Mark-sheets of undergraduate and postgraduate (qualifying) examinations.
(iii) Matriculation/Higher Secondary/Secondary School Certificate for verification of date of birth.
(iv) Certificate, in the case of SC/ST/OBC/PH category.

(v) Fellowship/Scholarship/UGC-NET award letter.

(b) After the submission/verification of the above documents/ certificates, to obtain a slip from the BRS (MS) office for depositing the required fees with the University through online and submit a copy of the fee receipt paid along with the joining report.

He may further note that,

(i) Teachers from a Department/Constituent College of the University of Delhi are exempted from payment of tuition fees on production of employment certificate.

(ii) The Ph.D. students have to pay fees annually in advance and if they fail to do so their names will be struck off from the rolls.

[redacted]

CHAIRPERSON

Encl: Joining Report Proforma

Mr. Brij Mohan
30/5A, 2nd Floor, Backside,
West Patel Nagar
New Delhi- 110008
Email- brijmohan6414@gmail.com

Copy forwarded for information and necessary action to:

1. The Head, Department of Mathematics, University of Delhi, Delhi-110007.
2. Dr. Raj Kumar (Supervisor), Kirori Mal College, **University of Delhi, University Enclave, Delhi- 110007**.
3. Prof. V. Ambethkar, Department of Mathematics, University of Delhi, Delhi, Member Advisory Committee.
4. Dr. Surendra Kumar, Department of Mathematics, University of Delhi, Delhi Member Advisory Committee.
5. The Dean (Examination), University of Delhi, Delhi-110007.

[redacted]

Section Officer

List of Publications

Published Research Papers:

1. **Brij Mohan**, Sachin Kumar and Raj Kumar: On investigation of kink-solitons and rogue waves to a new integrable (3+1)-dimensional KdV-type generalized equation in nonlinear sciences, Nonlinear Dynamics, 113:10261-10276 (2024). **Indexing**: SCIE, Scopus, MCQ: 0.81 (2023), IF: 5.6, SJR: 1.201 (Q1).

2. **Brij Mohan**, Sachin Kumar: Generalization and analytic exploration of soliton solutions for nonlinear evolution equations via a novel symbolic approach in fluids and nonlinear sciences, Chinese Journal of Physics, 92:10-21 (2024). **Indexing**: SCIE, Scopus, MCQ: 0.27 (2024), IF: 4.6, SJR: 0.587 (Q2).

3. **Brij Mohan**, Sachin Kumar and Raj Kumar: Higher-order rogue waves and dispersive solitons of a novel P-type (3+1)-D evolution equation in soliton theory and nonlinear waves, Nonlinear Dynamics, 111:20275-20288 (2023). **Indexing**: SCIE, Scopus, MCQ: 0.81 (2023), IF: 5.6, SJR: 1.201 (Q1).

4. Sachin Kumar, **Brij Mohan**: A novel analysis of Cole-Hopf transformations in different dimensions, solitons, and rogue waves for a (2+1)-dimensional shallow water wave equation of ion-acoustic waves in plasmas, Physics of Fluids, 35:127128 (2023). **Indexing**: SCIE, Scopus, MCQ: 0.21 (2011), IF: 4.6, SJR: 0.900 (Q1).

5. **Brij Mohan**, Sachin Kumar: Rogue-wave structures for a generalized (3+1)-dimensional nonlinear wave equation in liquid with gas bubbles, Physica Scripta, 99:105291 (2024). **Indexing**: SCIE, Scopus, MCQ: 0.08 (2013), IF: 2.6, SJR: 0.388 (Q2).

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF DELHI

UNDERTAKING FROM THE CO-AUTHOR(S) OF PAPERS OTHER THAN SUPERVISOR(S) WHICH ARE PART OF Ph.D. THESIS

Details of the Research Scholar and his/her supervisor(s)	
NAME OF THE RESEARCH SCHOLAR:	**BRIJ MOHAN**
DATE OF REGISTRATION:	**07-02-2022**
DATE OF PRE-Ph.D. SEMINAR:	
NAME OF THE SUPERVISOR(S):	**Prof. Raj Kumar**
Details of the co-author of giving this undertaking and the details of the publications written jointly with the co-author.	
NAME OF THE CO-AUTHOR	**Prof. Sachin Kumar**
IS HE/SHE A RESEARCH SCHOLAR/ FACULTY MEMBER? (Give details, including address)	Professor, Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi, India.
LIST OF RESEARCH PAPERS JOINTLY WRITTEN WITH THIS CO-AUTHOR *(Attach first pages of the reprint(s); For each paper, provide name of the authors in the same order as appeared in the paper, title of the paper, volume, year, issue no., pages, name and country of the publisher of the journal.)*	
1. **Brij Mohan, Sachin Kumar, Raj Kumar.** *On investigation of kink-solitons and rogue waves to a new integrable (3+1)-dimensional KdV-type generalized equation in nonlinear sciences*, 113, 10261–10276, (2024), Nonlinear Dynamics, Springer, United Kingdom (UK). **DOI:** 10.1007/s11071-024-10792-8 2. **Brij Mohan, Sachin Kumar.** *Generalization and analytic exploration of soliton solutions for nonlinear evolution equations via a novel symbolic approach in fluids and nonlinear sciences*, 92:10-21, (2024), Chinese Journal of Physics, Elsevier, Netherlands. **DOI:** 10.1016/j.cjph.2024.09.004 3. **Brij Mohan, Sachin Kumar, Raj Kumar.** *Higher-order rogue waves and dispersive solitons of a novel P-type (3+1)-D evolution equation in soliton theory and nonlinear waves*, 111, 20275–20288, (2023), Nonlinear Dynamics, Springer, United Kingdom (UK). **DOI:** 10.1007/s11071-023-08938-1 4. **Sachin Kumar, Brij Mohan.** *A novel analysis of Cole-Hopf transformations in different dimensions, solitons, and rogue waves for a (2+1)-dimensional shallow water wave equation of ion-acoustic waves in plasmas*, 35, 127128, (2023), Physics of Fluids, AIP Publishing, USA. **DOI:** 10.1063/5.0185772 5. **Brij Mohan, Sachin Kumar.** *Rogue-wave structures for a generalized (3+1)-dimensional nonlinear wave equation in liquid with gas bubbles*, 99, 105291, (2024), Physica Scripta, IOP Publishing, United Kingdom (UK). **DOI:** 10.1088/1402-4896/ad7cd9	
I have no objection if the above papers are fully or partially included in the Ph.D. thesis to be submitted in the Department of Mathematics, University of Delhi, by the above-named research scholar. I further undertake that I will not use the above paper(s) in full or partially for the award of any degree or diploma in this university or elsewhere.	
Date: 05/08/2025 Place: Delhi	[redacted] Signature of the co-author

Nonlinear Dyn (2025) 113:10261–10276
https://doi.org/10.1007/s11071-024-10792-8

RESEARCH

On investigation of kink-solitons and rogue waves to a new integrable (3+1)-dimensional KdV-type generalized equation in nonlinear sciences

Brij Mohan · Sachin Kumar · Raj Kumar

Received: 20 June 2024 / Accepted: 12 December 2024 / Published online: 19 December 2024

Abstract This research study proposes a novel (3+1)-dimensional Painlevé integrable KdV-type equation that generalizes well-known equations in soliton theory and nonlinear sciences. It illustrates the Painlevé analysis to establish the complete integrability of the proposed equation. We employ the Cole-Hopf transformations to get the bilinear equation in an auxiliary function and further construct it into Hirota's bilinear form. Utilizing the Hirota bilinear technique, we obtain the soliton solutions of kink types and their interactions up to the third order. It examines the rogue waves of the higher order using a direct symbolic approach up to the third order. For constructing the rogue waves, we transform the investigated equation from (3+1)-dimensional to a (1+1)-dimensional partial differential equation and form its Hirota bilinear form in transformed variables. It demonstrates the dynamics for the obtained kink-soliton and rogue wave solutions with appropriate parameter values using the symbolic system *Mathematica*. The interaction solutions of rogue waves show the dominating nature of more giant waves over smaller waves. We analyze the rogue dynamics in both the transformed and original variables. Solitons as solitary waves and rogue waves as extreme or monster waves are alluring concepts in various fields of nonlinear sciences, including oceanography, optical fibers, plasma physics, dynamical systems, and engineering.

Keywords Bilinear form · Painlevé analysis · Interaction solutions · Hirota bilinear technique · Direct symbolic approach

1 Introduction

In soliton theory and nonlinear sciences, solitons [1–7] have attracted researchers and scientists as a fascinating wave phenomenon. Having an equilibrium between dispersion and nonlinearity, they are distinguished by their ability to preserve their form and stability across vast distances. Solitons are essential for high-speed fiber optic communication systems called optical solitons or soliton pulses. By adjusting for the material's nonlinearity and the medium's dispersion, soliton transmission over extended distances can occur without significant distortion. This feature is necessary for reliable and efficient data transport in optical communication networks. Dispersive solitary waves or solitons are helpful in wave energy conversion, oceanography, and coastal engineering. Understanding and con-

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trolling these solitons can help to prevent coastal erosion, enhance wave prediction, and maximize wave energy extraction. The combination of the plasma's dispersive properties and the nonlinearity caused by particle interactions results in these solitons. They are helpful in many areas of plasma physics study, including fusion investigations, plasma heating, and wave propagation in magnetized plasma. Solitons are essential to understand and use in various domains, from optical communication and high-speed data transfer to coastal management and quantum technology development.

Localized substantial solitary waves in space-time, often known as rogue waves [8–14], possess a significant amplitude. These unpredictable waves have the potential to harm humans seriously. The evolution of rogue waves is a topic of great interest to many experts from different fields of nonlinear sciences. Rogues are more significant than the surrounding waves, making them noticeable due to their unusually high height. Nonlinear science studies rogue waves due to its contradiction of models for linear waves. The research on rogue waves predicts their occurrence and understands their underlying physics. The singularity in rogue wave solutions is a critical feature of their dynamics. Singularities occur when wave amplitudes theoretically approach infinity within finite time, corresponding to extreme and sudden amplification. This behavior is often linked to constructive interference among wave components and the inherent nonlinearity of the system, leading to "wave focusing." Techniques such as Painlevé analysis are applied to the governing equations to analyze these singularities. Such tools help to identify the locations and conditions under which singularities arise, offering insights into the mechanisms behind rogue wave formation. Phase shifts and arbitrary parameters in the solutions of nonlinear PDEs shape the structure and behavior of rogue waves; such solutions are called singular-like solutions. Adjusting these parameters allows us to simulate how energy concentrates into singular points, influencing rogue waves' peak height and transiency. Researchers observe similar phenomena by understanding these dynamics, essential for practical applications ranging from oceanography and meteorology to optics and quantum mechanics.

The improvement of maritime safety is one significant usage. One can use prediction models or algorithms to provide earlier observation and awareness systems to prevent harm caused by rogue waves. The maritime sector, gas or oil outlets, and infrastructure near the coast could all benefit from knowing this information. Therefore, we may attain greater functional security and affordable solutions by comprehending the dynamical analysis of building safe structures and designing strategies to reduce the impact of rogues. The study of solitons and rogue waves has garnered significant attention in recent decades due to their fascinating properties and broad applications in oceanography, optical fibers, plasma physics, and engineering fields. Various soliton equations, such as the (1+1)-dimensional Korteweg-de Vries (KdV) and the (2+1)-dimensional Kadomtsev-Petviashvili (KP) equations, have been widely explored for their ability to model nonlinear wave phenomena. However, these lower-dimensional systems often need to capture the complexities of higher-dimensional dynamics in real-world applications. Moreover, while many methods exist to verify the integrability and solvability of soliton equations, higher-order soliton and rogue wave interactions remain under explored, particularly in high-dimensional settings, and still need to be explored. Furthermore, studies on rogue wave dynamics contribute to our understanding of problematic situations, interactions of waves, and the emergence of more significant occurrences in several nonlinear phenomena.

In this research, we propose and investigate an (3+1)-D KdV-type generalized nonlinear equation as

$$u_{xxxy} + \sigma_1 u_{yt} + \sigma_2 (u_x u_y)_x + \sigma_3 u_{xx} + \sigma_4 u_{zz} = 0, \quad (1)$$

where $\sigma_{i=1,2,3,4}$ are as real parameters, and generalizes well-known equations:

– (3+1)-D Hirota bilinear equation [15] with $\sigma_1 = -1, \sigma_2 = 3, \sigma_3 = 0, \sigma_4 = -3$

$$u_{xxxy} - u_{yt} + 3(u_x u_y)_x - 3u_{zz} = 0, \quad (2)$$

– (3+1)-D Jimbo-Miwa equation [16] with $\sigma_1 = 2, \sigma_2 = 3, \sigma_3 = 0, \sigma_4 = -3$

$$u_{xxxy} + 2u_{yt} + 3(u_x u_y)_x - 3u_{zz} = 0, \quad (3)$$

– (2+1)-D BLMP equation [17] with $\sigma_1 = 1, \sigma_2 = -3, \sigma_3 = \sigma_4 = 0$

$$u_{xxxy} + u_{yt} - 3(u_x u_y)_x = 0, \quad (4)$$

– (2+1)-D KP equation [18] under the transformation $y \to x, z \to y, u_x \to u$ with $\sigma_1 = 1, \sigma_2 = 6, \sigma_3 = 0, \sigma_4 = \pm 3$

$$(u_t + 6uu_x + u_{xxx})_x \pm 3u_{yy} = 0, \tag{5}$$

– (1+1)-D KdV equation [19] under the transformation $y \to x, z \to x, u_x \to u$ with $\sigma_1 = 1, \sigma_2 = 6, \sigma_3 = \sigma_4 = 0$

$$u_t + 6uu_x + u_{xxx} = 0, \tag{6}$$

Localized solutions, including soliton, lump, breather, and others, are carried by the integrable evolution equation in specific directions. Analyzing the integrability of the nonlinear PDEs might result in exact and analytical solutions. The Painlevé test [20–22] can be used to verify the complete integrability of a nonlinear PDE. Finding out if a PDE can pass the test of Painlevé analysis gets somewhat tedious. However, symbolic systems make this investigation possible, like the system software *Mathematica* and *Matlab*. We look for particular explanations to understand the peculiarities of several facts accurately in various disciplines of nonlinear science. As mentioned previously, NLPDE has drawn the attention of several scholars to its ability to provide a wide range of solutions and closely simulate real-world scenarios. The dynamic analysis of rogue wave behavior resulting from nonlinear PDEs has made it an attractive research field for highlighting basic principles in water engineering, plasma, nonlinear sciences, and shallow water waves. Compared to existing literature, this work stands out by extending the dimension of the equation and offering a new approach to analyzing rogue wave interactions. While previous studies have mainly focused on lower-dimensional soliton solutions and first-order rogue waves, this research pushes the boundaries by considering higher-dimensional systems and providing third-order solutions. The symbolic computations carried out using Mathematica further enhance the precision and applicability of the solutions, making this method both efficient and effective for studying complex nonlinear wave phenomena. This efficiency and effectiveness make the method practical for current research and suggest its potential for further exploration and application in other nonlinear systems.

Nonlinear PDEs [23–31] deals with nonlinear functions used as models for complicated physical systems in various scientific domains. They are challenging to examine since no general analysis technique exists. Usually, each equation needs to be examined independently as a problem. Nonetheless, there are some circumstances in which broad approaches are appropriate. These techniques discretize the problem into a smaller grid after which they estimate the solution using mathematical procedures. Several methods are being used to obtain the analytic and exact solutions, such as the Darboux transformation [32–34]; the simplified Hirota's technique [35,36]; the Bäcklund transformation [37,38]; the Bilinear Neural Network Method [39,40]; Lie symmetry analysis [41–43]; the Hirota's bilinearization technique [44–47]; and others.

The manuscript is structured as follows: The following section investigates the Painlevé integrability of the proposed KdV-type nonlinear equation. In Sect. 3, we construct Hirota's bilinear form using the Cole-Hopf transformation and obtain the soliton solutions up to third order and depict the dynamics of the these solutions. Section 4 constructs the rogue wave solutions utilizing a direct symbolic approach with the bilinear form of the equation in transformed variables. It finds the rogue waves up to the third order and plots the dynamical structures for the obtained rogue solutions. In Sect. 5, we discuss the findings concerning the dynamic behaviors of the shown graphics, and the last Section concludes the remarks of our work and highlight its future scope.

2 Integrability: Painlevé analysis

A reliable method for analyzing the integrability of nonlinear PDEs is the Painlevé test. The primary goal of this analysis is to find movable singularity-free solutions for a nonlinear PDE. If a PDE passes the Painlevé test, it is considered P-integrable, indicating that complex structures can be solved using specialized functions. By verifying integrable conditions, Weiss et al. [48] provided the Painlev'e test to assess the integrability of the nonlinear PDEs. Three steps make up this analysis: first, it looks at the leading-order analysis; second, it finds the resonances; and third, it verifies the resonance conditions completely. If the simple poles of the solutions correspond to all moveable singularities, then the test is considered P-integrable. The field u is expanded by Laurent's series about the singular manifold $g = 0$ of an analytical function g as

$$u=\sum_{\lambda=0}^{\infty}u_\lambda g^{\lambda+\Lambda}, \tag{7}$$

where Λ and u_λ; are integer and arbitrary functions, respectively. On substitution of Eq. (7) in (1), with leading order analysis, we get

$$\Lambda=-1,$$

with

$$u_0=\frac{6g_x}{\alpha_2}.$$

It gets the resonances as

$$\lambda=-1,\quad 1,\quad 4,\quad 6.$$

The resonance $\lambda=-1$ shows the arbitrary choice for singular manifold $g=0$. The analysis finds the functions u_λ explicitly for $\lambda=0,2,3,5$ and as arbitrary for positive resonances. The positive resonances satisfied the compatibility conditions. Thus, the investigated KdV-type equation is Painlevé integrable.

3 Bilinear form and *N*-soliton solutions

We take Φ_i as the phase in the Eq. (1) as

$$\Phi_i=p_i x+q_i y+r_i z-w_i t, \tag{8}$$

with w_i as dispersions and p_i, q_i, r_i real parameters. Putting $u=\mathrm{e}^{\Phi_i}$ into Eq. (1) for linear terms, we get the dispersion as

$$w_i=\frac{\alpha_3 p_i^2+p_i^3 q_i+\alpha_4 r_i^2}{\alpha_1 q_i}. \tag{9}$$

Considering the Cole-Hopf transformation of auxiliary function f as

$$u=P(\log f)_x, \tag{10}$$

and puting with $f(U,V)=1+\mathrm{e}^{\Phi_1}$ and Eq. (9) into Eq. (1). On solving for P, we get

$$P=\frac{6}{\alpha_2}.$$

Now, we can transform the Eq. (1) with Eq. (10) in f as

$$\begin{aligned}&ff_{xxxy}-3f_xf_{xxy}+3f_{xx}f_{xy}-f_{xxx}f_y\\&+\alpha_1(ff_{yt}-f_tf_y)+\alpha_3(ff_{xx}-f_x^2)\\&+\alpha_4(ff_{zz}-f_z^2)=0\end{aligned} \tag{11}$$

that is a bilinear equation and can be shown in Hirota's bilinear form. Hirota [19] designed the differential operators $D_k : k=x,y,z,t$ as

$$\begin{aligned}&D_x^{r_1}D_y^{r_2}D_z^{r_3}D_t^{r_4}U(x,y,z,t)V(x,y,z,t)\\&=\left(\frac{\partial}{\partial_x}-\frac{\partial}{\partial_{x'}}\right)^{r_1}\left(\frac{\partial}{\partial_y}-\frac{\partial}{\partial_{y'}}\right)^{r_2}\\&\left(\frac{\partial}{\partial_z}-\frac{\partial}{\partial_{z'}}\right)^{r_3}\left(\frac{\partial}{\partial_t}-\frac{\partial}{\partial_{t'}}\right)^{r_4}\\&U(x,y,z,t)V(x',y',z',t')|_{x=x',y=y',z=z',t=t'},\end{aligned}$$

with x', y', z', t' as formal variables and and $r_i : 1\le i\le 4$ as the positive integers.

Thus, the Eq. (11) has its Hirota's bilinear form as

$$\left[D_x^3D_y+\alpha_1D_yD_t+\alpha_3D_x^2+\alpha_4D_z^2\right]f.f=0. \tag{12}$$

We obtain the N-soliton solution by considering an expression for the function f in closed-form given by Hirota as

$$f=\sum_{\eta=0,1}\exp\left(\sum_{i=1}^{N}\eta_i\Phi_i+\sum_{1=i<j}^{N}A_{ij}\eta_i\eta_j\right), \tag{13}$$

where $\sum_{\eta=0,1}$ indicates the summation of all possible combinations for $\eta_i=0,1$ for $1\le i\le N$.

3.1 Single kink-soliton

For $N=1$ in Eq. (13), we have $\eta_1=0$ and 1 so we take f as

$$f=f_1=1+\mathrm{e}^{\Phi_1}=1+\mathrm{e}^{p_1x+q_1y+r_1z-w_1t}, \tag{14}$$

which satisfied the Eq. (12). Thus, on substituting (14) with its derivative in the Eq. (10), we get 1-soliton solution

$$u=u_1=\frac{6p_1e^{p_1x+q_1y+r_1z}}{\alpha_2\left(\exp\left(\frac{t(\alpha_3p_1^2+p_1^3q_1+\alpha_4r_1^2)}{\alpha_1q_1}\right)+e^{p_1x+q_1y+r_1z}\right)}, \tag{15}$$

3.2 Two kink-solitons

Having $N = 2$ in the Eq. (13), we have $\eta_1 = \eta_2 = 0, 1$. So there will be four combinations of $\{\eta_1, \eta_2\}$ as $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$, therefore, the function f is

$$\begin{aligned} f = f_2 &= 1 + e^{\Phi_1} + e^{\Phi_2} + e^{A_{12}+\Phi_1+\Phi_2} \\ &= 1 + e^{\Phi_1} + e^{\Phi_2} + a_{12}e^{\Phi_1+\Phi_2}, \end{aligned} \tag{16}$$

where $a_{12} = e^{A_{12}}$.

On substituting the Eq. (16) into Eq. (12), we get

$$a_{12} = \frac{p_1^2 q_2\left(\alpha_3 q_2 - 3p_2 q_1(q_1 - q_2)\right) + p_2 p_1 q_1 q_2\left(3p_2(q_1 - q_2) - 2\alpha_3\right) + \alpha_3 p_2^2 q_1^2 + \alpha_4 (q_2 r_1 - q_1 r_2)^2}{p_1^2 q_2\left(\alpha_3 q_2 - 3p_2 q_1(q_1 + q_2)\right) - p_2 p_1 q_1 q_2\left(3p_2(q_1 + q_2) + 2\alpha_3\right) + \alpha_3 p_2^2 q_1^2 + \alpha_4 (q_2 r_1 - q_1 r_2)^2} \tag{17}$$

Thus, by putting Eq. (16) with (17) into (10), gives a 2-soliton solution for Eq. (1) as

$$u = u_2 = \frac{6}{\alpha_2}(\log f_2)_x \tag{18}$$

3.3 Three kink-solitons

For $N = 3$ in Eq. (13), we have $\eta_1, \eta_2, \eta_3 = 0, 1$ so the total combinations for $\{\eta_1, \eta_2, \eta_3\}$ will be eight as $\{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$, therefore, the function f is as

$$\begin{aligned} f = f_3 = 1 &+ e^{\Phi_1} + e^{\Phi_2} + e^{\Phi_3} + a_{12}e^{\Phi_1+\Phi_2} \\ &+ a_{13}e^{\Phi_1+\Phi_3} \\ &+ a_{23}e^{\Phi_2+\Phi_3} + b_{123}e^{\Phi_1+\Phi_2+\Phi_3}, \end{aligned} \tag{19}$$

where $a_{ij} = e^{A_{ij}}$ and $b_{123} = e^{A_{12}+A_{13}+A_{23}} = a_{12} + a_{13} + a_{23}$.

The Eq. (17) can be generalized as

$$a_{ij} = \frac{p_i^2 q_j\left(\alpha_3 q_j - 3q_i p_j\left(q_i - q_j\right)\right) + p_i q_i p_j q_j\left(3p_j\left(q_i - q_j\right) - 2\alpha_3\right) + \alpha_3 q_i^2 p_j^2 + \alpha_4\left(r_i q_j - q_i r_j\right)^2}{p_i^2 q_j\left(\alpha_3 q_j - 3q_i p_j\left(q_i + q_j\right)\right) - p_i q_i p_j q_j\left(3p_j\left(q_i + q_j\right) + 2\alpha_3\right) + \alpha_3 q_i^2 p_j^2 + \alpha_4\left(r_i q_j - q_i r_j\right)^2}, \tag{20}$$

for the auxiliary function $f = 1 + e^{\Phi_1} + e^{\Phi_2} + a_{ij}e^{\Phi_1+\Phi_2}$; $1 = i < j = 3$. By substituting the Eq. (19) with (20) into the Eq. (10), we obtain the 3-soliton solution as

$$u = u_3 = \frac{6}{\alpha_2}(\log f_3)_x \tag{21}$$

4 Bilinear form and rogue waves

We consider the transformations $u = u(\xi, \eta)$ with $\xi = x+t$ and $\eta = y+z$ in Eq. (1). Thus, we get transformed equation as

$$\begin{aligned} &\alpha_2 u_\eta u_{\xi\xi} + \alpha_4 u_{\eta\eta} + \alpha_2 u_\xi u_{\xi\eta} + \alpha_1 u_{\xi\eta} \\ &\quad + \alpha_3 u_{\xi\xi} + u_{\xi\xi\xi\eta} = 0. \end{aligned} \tag{22}$$

Taking the phase Φ_i in Eq. (22)

$$\Phi_i = p_i\xi - w_i\eta, \tag{23}$$

having w_i as dispersions and p_i as real-parameter. Putting $u(\xi, \eta) = e^{\Phi_i}$ into the Eq. (22), with linear terms, get

$$w_i = \frac{\alpha_1 p_i + p_i^3 \pm p_i\sqrt{\alpha_1^2 - 4\alpha_3\alpha_4 + 2\alpha_1 p_i^2 + p_i^4}}{2\alpha_4}. \tag{24}$$

Considering the logarithmic transformation

$$u(\xi, \eta) = P(\log f)_\xi, \tag{25}$$

with P as constant and f as auxiliary function. Putting the transformation (25) with $f(\xi, \eta) = 1 + e^{\Phi_1}$ into Eq. (22) gives

$$P = \frac{6}{\alpha_2}.$$

So, the transformation (25) gives a biilinear equation in f of the Eq. (22) as

$$\begin{aligned} &f f_{\xi\xi\xi\eta} - 3f_\xi f_{\xi\xi\eta} + 3f_{\xi\eta}f_{\xi\xi} - f_\eta f_{\xi\xi\xi} \\ &\quad + \alpha_1(f f_{\xi\eta} - f_\eta f_\xi) \\ &\quad + \alpha_3(f f_{\xi\xi} - f_\xi^2) + \alpha_4 \end{aligned}$$

$$(f f_{\eta\eta} - f_\eta^2) = 0. \tag{26}$$

Using Hirota's differential operators $D_i : i = \xi, \eta$

$$D_\xi^{n_1} D_\eta^{n_2} f(\xi, \eta) g(\xi, \eta) = \left(\frac{\partial}{\partial_\xi} - \frac{\partial}{\partial_{\xi'}}\right)^{n_1} \left(\frac{\partial}{\partial_\eta} - \frac{\partial}{\partial_{\eta'}}\right)^{n_2} f(\xi, \eta) g(\xi', \eta')|_{\xi=\xi', \eta=\eta'},$$

with ξ', η' as formal variables and and $n_i : i = 1, 2$ as positive integers, the Eq. (26) has its Hirota's bilinear form as

$$\left[D_\xi^3 D_\eta + \alpha_1 D_\xi D_\eta + \alpha_3 D_\xi^2 + \alpha_4 D_\eta^2\right] f.f = 0, \tag{27}$$

which shows the similar pattern for D-operators to the bilinear Eq. (12) in original variables x, y, z, t.

We obtain the rogue waves solutions by considering the function f [49,50] as

$$f(\xi, \eta) = \sum_{k=0}^{\frac{n(n+1)}{2}} \sum_{j=0}^{k} c_{n(n+1)-2k, 2j} \xi^{n(n+1)-2k} \eta^{2j}, \tag{28}$$

where n and $c_{r,s}; r, s \in \{0, 2, \cdots, k(k+1)\}$ are positive integer and the constants, respectively.

4.1 First-order rogue waves

For $n = 1$ in Eq. (28), we take auxiliary function $f(\xi, \eta)$ as

$$f = f_1 = c_{2,0}\xi^2 + c_{0,2}\eta^2 + c_{0,0}. \tag{29}$$

On having Eq. (29) into the Eq. (27), and equating zero the coefficients of distinct powers of $\xi^r \eta^s; r, s \in Z$, we obtain a system of equations

$$\begin{aligned} 2\alpha_4 c_{0,0} c_{0,2} + 2\alpha_3 c_{0,0} c_{2,0} &= 0, \\ 2\alpha_4 c_{0,2} c_{2,0} - 2\alpha_3 c_{2,0}^2 &= 0, \\ 2\alpha_3 c_{0,2} c_{2,0} - 2\alpha_4 c_{0,2}^2 &= 0. \end{aligned} \tag{30}$$

On solving above system, we get parameter values as

$$c_{0,0} = 0, \quad c_{0,2} = \frac{\alpha_4 c_{0,2}}{\alpha_3}, \quad c_{2,0} = c_{2,0}. \tag{31}$$

Thus, the function f in (29) becomes

$$f = f_1 = c_{0,2}\left(\frac{\alpha_4 \xi^2}{\alpha_3} + \eta^2\right). \tag{32}$$

On substituting the Eq. (32) into (25), we get a solution of 1^{st}-order rogue waves as

$$u(\xi, \eta) = u_1 = \frac{12\alpha_4 \xi}{\alpha_2\left(\alpha_3 \eta^2 + \alpha_4 \xi^2\right)}. \tag{33}$$

4.2 Second-order rogue waves

For 2^{nd}-order rogue waves, we take f for $n = 2$ in Eq. (28) as

$$\begin{aligned} f = f_2 = {} & c_{6,0}\xi^6 + c_{4,2}\xi^4\eta^2 + c_{4,0}\xi^4 \\ & + c_{2,4}\xi^2\eta^4 + c_{2,2}\xi^2\eta^2 + c_{2,0}\xi^2 \\ & + c_{0,6}\eta^6 + c_{0,4}\eta^4 + c_{0,2}\eta^2 + c_{0,0}. \end{aligned} \tag{34}$$

Substituting Eq. (34) into the Eq. (27), and equating zero the coefficients of distinct powers of $\xi^r \eta^s; r, s \in Z$, gives a system. On solving the system, we get values

$$\begin{aligned} & c_{0,0} = \frac{29\alpha_4 c_{4,2}}{2\alpha_1^3 \alpha_3}, \quad c_{0,2} = \frac{231 c_{4,2}}{4\alpha_1^2}, \\ & c_{0,4} = \frac{5\alpha_3 c_{4,2}}{\alpha_1 \alpha_4}, \quad c_{0,6} = \frac{\alpha_3^2 c_{4,2}}{3\alpha_4^2}, \\ & c_{2,0} = -\frac{9\alpha_4 c_{4,2}}{4\alpha_1^2 \alpha_3}, \quad c_{2,2} = \frac{12 c_{4,2}}{\alpha_1}, \\ & c_{2,4} = \frac{\alpha_3 c_{4,2}}{\alpha_4}, \quad c_{4,0} = -\frac{\alpha_4 c_{4,2}}{\alpha_1 \alpha_3}, \quad c_{4,2} = c_{4,2} \\ & c_{6,0} = \frac{\alpha_4 c_{4,2}}{3\alpha_3}. \end{aligned} \tag{35}$$

Thus, the function f in (34) becomes

$$\begin{aligned} f = f_2 = \frac{c_{4,2}}{12}\Bigg(& \frac{12\alpha_3 \eta^4 \xi^2}{\alpha_4} + \frac{144\eta^2\xi^2}{\alpha_1} \\ & + \frac{4\alpha_3^2 \eta^6}{\alpha_4^2} + \frac{60\alpha_3 \eta^4}{\alpha_1 \alpha_4} + \frac{693\eta^2}{\alpha_1^2} \\ & + \frac{4\alpha_4 \xi^6}{\alpha_3} - \frac{12\alpha_4 \xi^4}{\alpha_1 \alpha_3} - \frac{27\alpha_4 \xi^2}{\alpha_1^2 \alpha_3} \\ & + \frac{174\alpha_4}{\alpha_1^3 \alpha_3} + 12\eta^2\xi^4\Bigg). \end{aligned} \tag{36}$$

On putting Eq. (36) into (25), we get a solution for 2^{nd}-order rogue waves as

$$u(\xi,\eta)=u_2=\frac{36\xi\left(\frac{4\alpha_3\eta^4}{\alpha_4}+\frac{48\eta^2}{\alpha_1}-\frac{8\alpha_4\xi^2}{\alpha_1\alpha_3}+\frac{4\alpha_4\xi^4}{\alpha_3}-\frac{9\alpha_4}{\alpha_1^2\alpha_3}+8\eta^2\xi^2\right)}{\alpha_2\left(\frac{12\alpha_3\eta^4\xi^2}{\alpha_4}+\frac{144\eta^2\xi^2}{\alpha_1}+\frac{4\alpha_3^2\eta^6}{\alpha_4^2}+\frac{60\alpha_3\eta^4}{\alpha_1\alpha_4}+\frac{693\eta^2}{\alpha_1^2}+\frac{4\alpha_4\xi^6}{\alpha_3}-\frac{12\alpha_4\xi^4}{\alpha_1\alpha_3}-\frac{27\alpha_4\xi^2}{\alpha_1^2\alpha_3}+\frac{174\alpha_4}{\alpha_1^3\alpha_3}+12\eta^2\xi^4\right)}. \quad (37)$$

4.3 Third-order rogue waves

Take $n=3$ in Eq. (28), we get auxiliary function f as

$$\begin{aligned} f=f_3&=c_{12,0}\xi^{12}+c_{10,2}\xi^{10}\eta^2+c_{10,0}\xi^{10}\\ &+c_{8,4}\xi^8\eta^4+c_{8,2}\xi^8\eta^2\\ &+c_{8,0}\xi^8+c_{6,6}\xi^6\eta^6+c_{6,4}\xi^6\eta^4+c_{6,2}\xi^6\eta^2\\ &+c_{6,0}\xi^6+c_{4,8}\xi^4\eta^8+c_{4,6}\xi^4\eta^6\\ &+c_{4,4}\xi^4\eta^4+c_{4,2}\xi^4\eta^2+c_{4,0}\xi^4\\ &+c_{2,10}\xi^2\eta^{10}+c_{2,8}\xi^2\eta^8+c_{2,6}\xi^2\eta^6\\ &+c_{2,4}\xi^2\eta^4+c_{2,2}\xi^2\eta^2\\ &+c_{2,0}\xi^2+c_{0,12}\eta^{12}+c_{0,10}\eta^{10}+c_{0,8}\eta^8\\ &+c_{0,6}\eta^6+c_{0,4}\eta^4+c_{0,2}\eta^2+c_{0,0}. \end{aligned} \quad (38)$$

Substituting Eq. (38) into the Eq. (27), and equating zero the coefficients of distinct powers of $\xi^r\eta^s$; $r,s\in Z$, gives a system. On solving this system, we get values as

$$\begin{aligned} c_{0,0}&=\frac{7353680000\alpha_4c_{10,2}}{1113\alpha_1^6\alpha_3},\\ c_{0,2}&=-\frac{6077833600c_{10,2}}{371\alpha_1^5},\\ c_{0,4}&=-\frac{6359200\alpha_3c_{10,2}}{7\alpha_1^4\alpha_4},\ c_{0,6}=-\frac{800\alpha_3^2c_{10,2}}{3\alpha_1^3\alpha_4^2},\\ c_{0,8}&=\frac{160\alpha_3^3c_{10,2}}{\alpha_1^2\alpha_4^3},\ c_{0,10}=-\frac{10\alpha_3^4c_{10,2}}{\alpha_1\alpha_4^4},\\ c_{0,12}&=\frac{\alpha_3^5c_{10,2}}{6\alpha_4^5},\ c_{2,0}=\frac{72534400\alpha_4c_{10,2}}{371\alpha_1^5\alpha_3},\\ c_{2,2}&=\frac{1521600c_{10,2}}{7\alpha_1^4},\\ c_{2,4}&=-\frac{800\alpha_3c_{10,2}}{\alpha_1^3\alpha_4},\\ c_{2,6}&=\frac{560\alpha_3^2c_{10,2}}{\alpha_1^2\alpha_4^2},\ c_{2,8}=-\frac{30\alpha_3^3c_{10,2}}{\alpha_1\alpha_4^3}\\ c_{2,10}&=\frac{\alpha_3^4c_{10,2}}{\alpha_4^4},\ c_{4,0}=-\frac{55200\alpha_4c_{10,2}}{7\alpha_1^4\alpha_3},\\ c_{4,2}&=\frac{800c_{10,2}}{\alpha_1^3},\ c_{4,4}=\frac{800\alpha_3c_{10,2}}{\alpha_1^2\alpha_4},\\ c_{4,6}&=-\frac{20\alpha_3^2c_{10,2}}{\alpha_1\alpha_4^2},\\ c_{4,8}&=\frac{5\alpha_3^3c_{10,2}}{2\alpha_4^3},\ c_{6,0}=\frac{800\alpha_4c_{10,2}}{3\alpha_1^3\alpha_3},\\ c_{6,2}&=\frac{560c_{10,2}}{\alpha_1^2},\ c_{6,4}=\frac{20\alpha_3c_{10,2}}{\alpha_1\alpha_4},\\ c_{6,6}&=\frac{10\alpha_3^2c_{10,2}}{3\alpha_4^2},\\ c_{8,0}&=\frac{160\alpha_4c_{10,2}}{\alpha_1^2\alpha_3},\\ c_{8,2}&=\frac{30c_{10,2}}{\alpha_1},\ c_{8,4}=\frac{5\alpha_3c_{10,2}}{2\alpha_4},\ c_{10,2}=c_{10,2},\\ c_{10,0}&=\frac{10\alpha_4c_{10,2}}{\alpha_1\alpha_3},\\ c_{12,0}&=\frac{\alpha_4c_{10,2}}{6\alpha_3}, \end{aligned} \quad (39)$$

with $a_{10,2}$ as an arbitrary parameter. Thus, the Eq. (29) becomes

$$\begin{aligned} f=f_3=\frac{c_{10,2}}{2226}\Bigg(&\frac{2226\alpha_3^4\eta^{10}\xi^2}{\alpha_4^4}+\frac{5565\alpha_3^3\eta^8\xi^4}{\alpha_4^3}\\ &-\frac{66780\alpha_3^3\eta^8\xi^2}{\alpha_1\alpha_4^3}\\ &+\frac{7420\alpha_3^2\eta^6\xi^6}{\alpha_4^2}-\frac{44520\alpha_3^2\eta^6\xi^4}{\alpha_1\alpha_4^2}\\ &+\frac{1246560\alpha_3^2\eta^6\xi^2}{\alpha_1^2\alpha_4^2}+\frac{5565\alpha_3\eta^4\xi^8}{\alpha_4}\\ &+\frac{44520\alpha_3\eta^4\xi^6}{\alpha_1\alpha_4}\\ &+\frac{1780800\alpha_3\eta^4\xi^4}{\alpha_1^2\alpha_4}-\frac{1780800\alpha_3\eta^4\xi^2}{\alpha_1^3\alpha_4} \end{aligned}$$

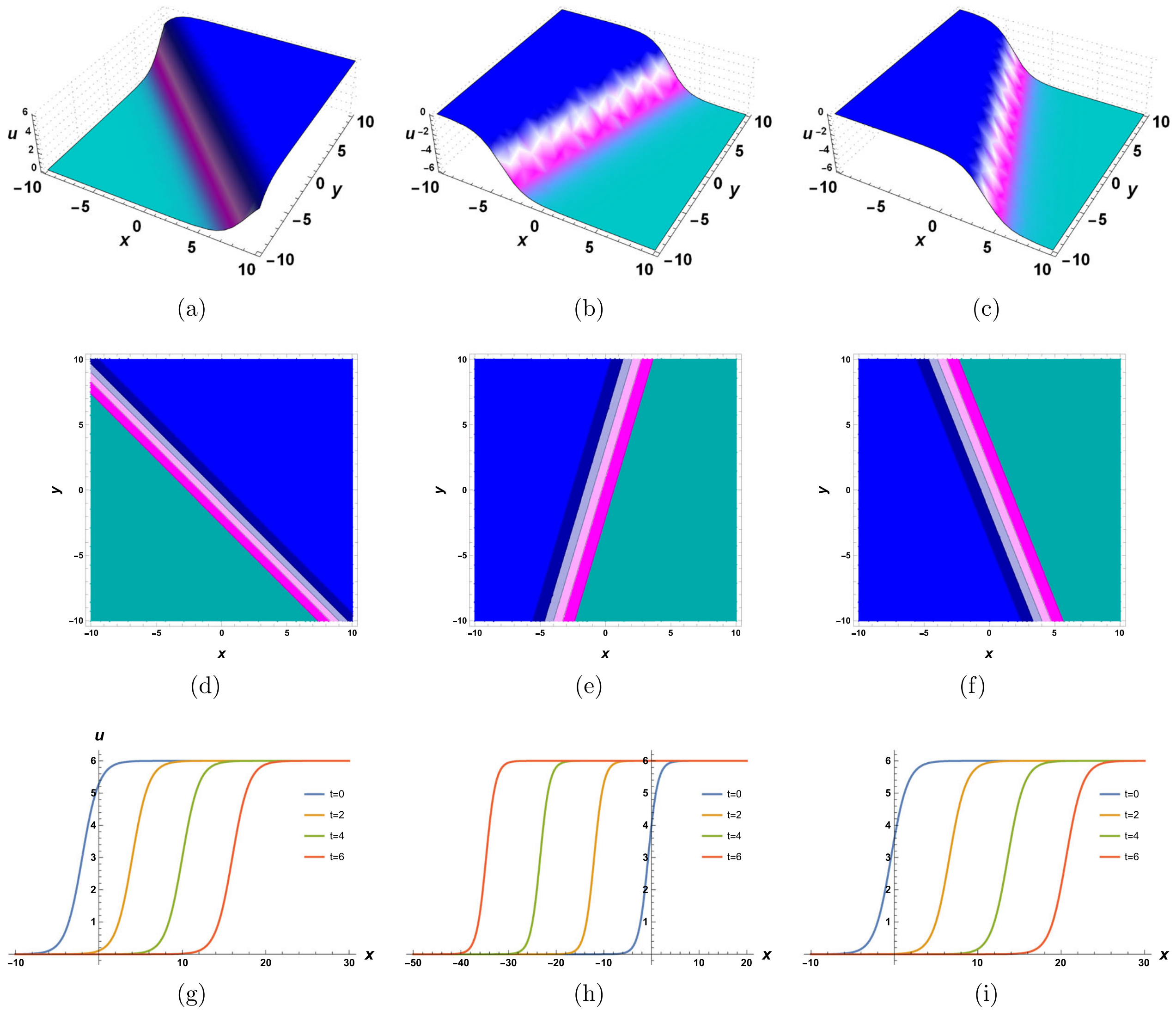

Fig. 1 Dynamics of single kink-soliton for (15) with $z = 0$. **d**, **e** and **g–i** depict the contour plots in xy-plane and 2D plots at different time t, respectively

$$
\begin{aligned}
&+\frac{66780\eta^2\xi^8}{\alpha_1} \\
&+\frac{1246560\eta^2\xi^6}{\alpha_1^2}+\frac{1780800\eta^2\xi^4}{\alpha_1^3} \\
&+\frac{483868800\eta^2\xi^2}{\alpha_1^4}+\frac{371\alpha_3^5\eta^{12}}{\alpha_4^5} \\
&-\frac{22260\alpha_3^4\eta^{10}}{\alpha_1\alpha_4^4}+\frac{356160\alpha_3^3\eta^8}{\alpha_1^2\alpha_4^3} \\
&-\frac{593600\alpha_3^2\eta^6}{\alpha_1^3\alpha_4^2}-\frac{2022225600\alpha_3\eta^4}{\alpha_1^4\alpha_4} \\
&-\frac{36467001600\eta^2}{\alpha_1^5}+\frac{371\alpha_4\xi^{12}}{\alpha_3} \\
&+\frac{22260\alpha_4\xi^{10}}{\alpha_1\alpha_3}+\frac{356160\alpha_4\xi^8}{\alpha_1^2\alpha_3} \\
&+\frac{593600\alpha_4\xi^6}{\alpha_1^3\alpha_3}+\frac{435206400\alpha_4\xi^2}{\alpha_1^5\alpha_3} \\
&-\frac{17553600\alpha_4\xi^4}{\alpha_1^4\alpha_3} \\
&\left.+\frac{14707360000\alpha_4}{\alpha_1^6\alpha_3}+2226\eta^2\xi^{10}\right). \qquad (40)
\end{aligned}
$$

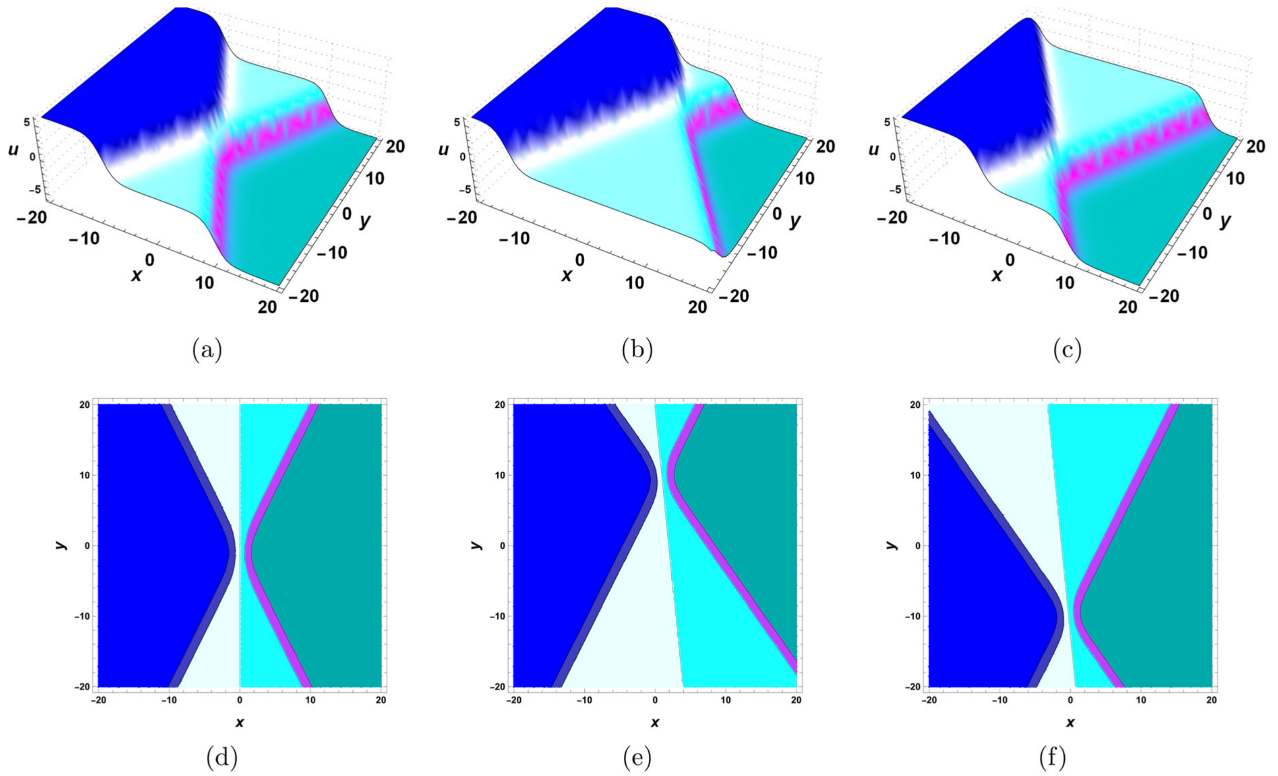

Fig. 2 Dynamics of two kink-solitons for (18) with $z = 0$. **d**, **e** depicts the contour plots for **a**–**c** in xy-plane

On having Eq. (40) into (25), we get a solution of 3^{rd}-order rogue waves as

$$u = u_3 = \frac{6}{\alpha_2}(\log f_3)_\xi. \tag{41}$$

5 Results and findings

The proposed KdV-type evolution equation showed the completely integrable using Painlevé analysis. Thus, it has soliton solutions for kink type with the Hirota bilinear technique. The first-order soliton solution generated the single kink-soliton, and the second and third-order soliton solutions showed the interaction solutions for two and three kink-solitons with an appropriate selection of parameters. After that, the rogue wave solutions for the investigated equation utilize a direct symbolic approach. The first-order rogue solution generated a single rogue wave solution, and second and third-order rogue solutions gave the interactions of rogue waves. The dynamics of rogue wave solutions have been shown in transformed variables ξ, η, and in the starting variables x, y, z, t in $\xi\eta$, xt, and xy planes. In this context, the dynamical findings are as follows:

- Figure 1 show the one solitons of kink-type, and the solitons (a) and (c) are propagating to the right, while (b) is propagating to the left of x-axis. The illustrated kink-solitons have the parameter values as **(a)** $p_1 = q_1 = r_1 = 1, \alpha_i = 1, 1 \le i \le 4$; **(b)** $p_1 = 1, q_1 = -0.3, r_1 = \alpha_1 = 1, \alpha_2 = -1, \alpha_3 = \alpha_4 = 1$; and **(c)** $p_1 = 1, q_1 = 0.4, r_1 = 0, \alpha_1 = 1, \alpha_2 = -1, \alpha_3 = \alpha_4 = 1$.
- In Fig. 2, we show the two-soliton solutions of kink-types. The dynamics depicts the interactions of the two kink-type soliton solutions with chosen parameter values. The showed kink-solitons have the parameter values as **(a)** $p_1 = 1, p_2 = -1, q_1 = 0.5 = q_2, r_1 = 0.5 = r_2, \alpha_1 = 1, \alpha_2 = -1, \alpha_3 = \alpha_4 = 1$; **(b)** $p_1 = 1, p_2 = -1, q_1 = 0.7, q_2 = 0.5, r_1 = 0.5 = r_2, \alpha_1 = 1, \alpha_2 = -1, \alpha_3 = \alpha_4 = 1$; and **(c)** $p_1 = 1, p_2 = -1, q_1 = 0.7, q_2 = 0.5, r_1 = 0.5 = r_2, \alpha_1 = 1, \alpha_2 = -1, \alpha_3 = \alpha_4 = 1$.
- Figure 3 depict the three-soliton solutions of kink-types. The dynamics depicts the interactions of

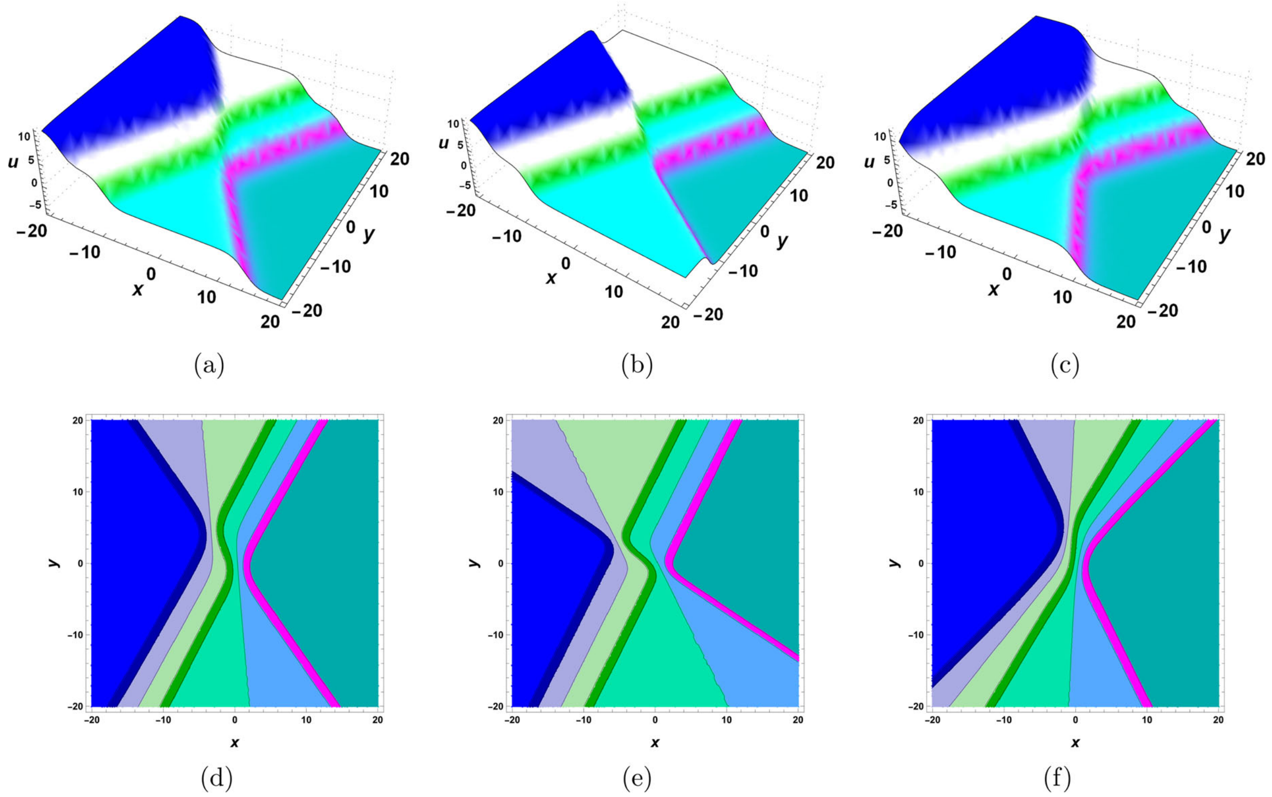

Fig. 3 Dynamics of three kink-solitons for (21) with $z = 0$. **d**, **e** depicts the contour plots for **a**–**c** in xy-plane

the three kink-type soliton solutions with chosen parameters. The showed kink-solitons have the parameter values as **(a)** $p_1 = 1, p_2 = -1 = p_3, q_1 = 0.7, q_2 = 0.5, q_3 = 0.6, r_1 = 0.5 = r_2 = r_3, \alpha_1 = 1, \alpha_2 = -1, \alpha_3 = \alpha_4 = 1$; **(b)** $p_1 = 1, p_2 = -1 = p_3, q_1 = 1.5, q_2 = 0.5 = q_3, r_1 = 0.5 = r_2, r_3 = 0.6, \alpha_1 = 1, \alpha_2 = -1, \alpha_3 = \alpha_4 = 1$; and **(c)** $p_1 = -1, p_2 = 1, p_3 = -1, q_1 = 0.7, q_2 = 0.5, q_3 = 0.6, r_1 = 0.5 = r_2 = r_3, \alpha_1 = 1, \alpha_2 = -1, \alpha_3 = \alpha_4 = 1$.

- In Fig. 4 and 5, we illustrate the rogue waves of first order which depict the single rogues with singularities at $\xi = \eta = 0$. For all three plots in Fig. 4, positive and negative direction of ξ shows the bright and the dark part of the rogue wave dynamics. Figure 5 depicts the single rogue wave structures in the original variables x, y, z, t. (a)-(c) shows the periodic behavior w.r.t. time variable in xt-plane with $y = z = 0$ and (d)-(f) shows the single rogue waves in xy-plane with $z = t = 0$. The showed single rogue waves for both figures have the parameter values as **(a)** $\alpha_2 = 5, \alpha_3 = \alpha_4 = 1$; **(b)** $\alpha_2 = 10, \alpha_3 = 5, \alpha_4 = 1$; and **(c)** $\alpha_2 = 1, \alpha_3 = 3, \alpha_4 = 5$.
- Figure 6 and 7 depict the rogue waves of second order which show the two rogues having dominating nature of extreme rogue waves to the smaller rogues. For all three graphs in Fig. 6, the two rogues intersect at $\xi = \eta = 0$ with having their bight and dark parts. Figure 7 shows the two rogue wave structures in the original variables x, y, z, t. (a)-(c) shows the periodic nature w.r.t. time variable, in xt-plane with $y = z = 0$ and (d)-(f) shows the two rogue waves in xy-plane with $z = t = 0$. The showed second-order rogue waves for both figures have the parameter values as **(a)** $\alpha_1 = 0.2, \alpha_2 = -0.8, \alpha_3 = \alpha_4 = 1$; **(b)** $\alpha_1 = 0.3, \alpha_2 = 0.2, \alpha_3 = 1, \alpha_4 = 2$; and **(c)** $\alpha_1 = 0.1, \alpha_2 = -0.5, \alpha_3 = -3, \alpha_4 = -2$.
- In Fig. 8 and 9, we illustrate the third-order rogue waves that depict the four rogue waves having dominating nature of extreme rogue waves to the smaller rogues. For all three plots in Fig. 8, the four rogues depict the intersections having their

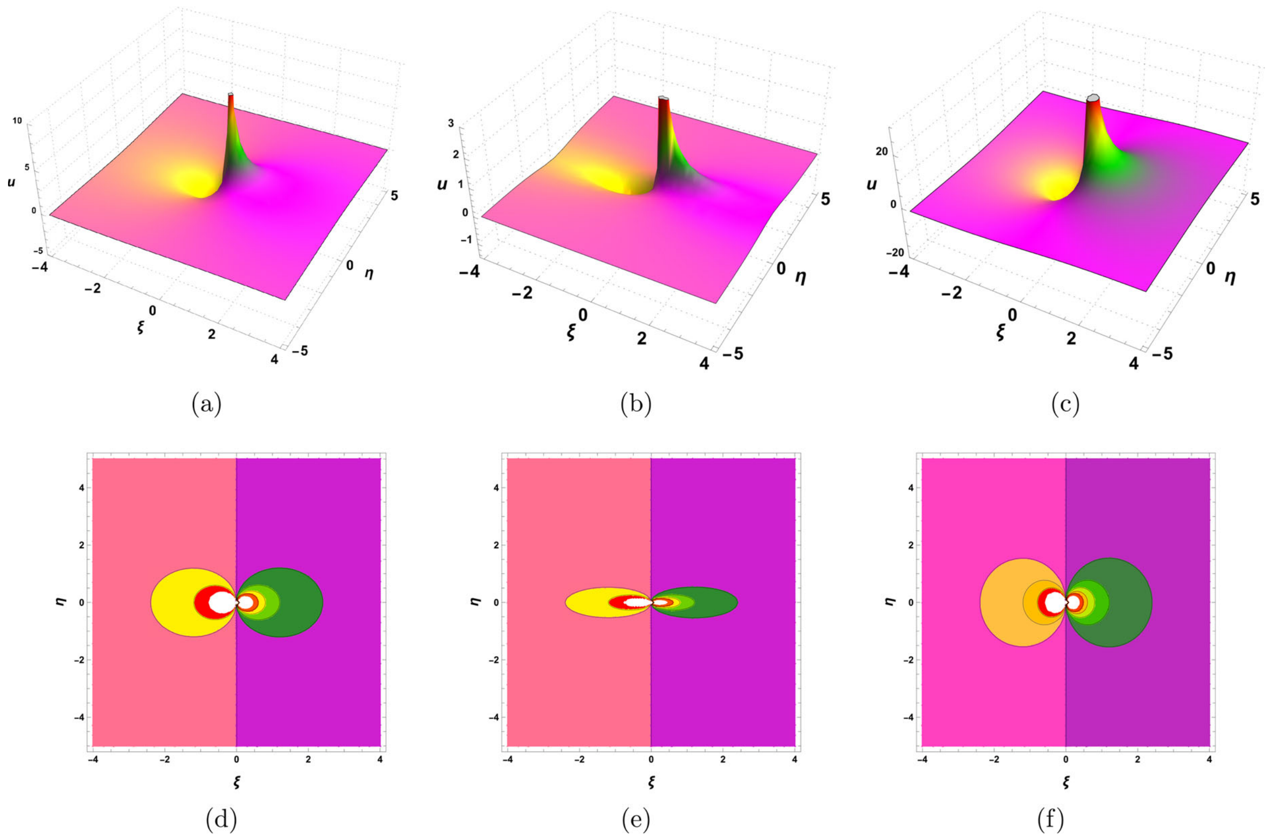

Fig. 4 Dynamics of first-order rogue waves for solution (33) in transformed variables ξ and η. **a**, **b** depicts the 3D profiles in $\xi\eta$-plane, and **d**–**f** shows the contour plots for the **a**–**c**

bight and dark parts. Fig. 9 shows the four rogue wave structures in the original variables x, y, z, t. (a)-(c) shows the periodicity w.r.t. time variable in xt-plane with $y = z = 0$ and (d)-(f) shows the four rogue waves in xy-plane with $z = t = 0$. The showed third-order rogue waves for both figures have the parameter values as **(a)** $\alpha_1 = -0.3, \alpha_2 = 0.1, \alpha_3 = -0.7, \alpha_4 = -0.5$; **(b)** $\alpha_1 = -0.6, \alpha_2 = -0.1, \alpha_3 = 0.1, \alpha_4 = 0.2$; and **(c)** $\alpha_1 = -0.5, \alpha_2 = -0.05, \alpha_3 = 0.4, \alpha_4 = 0.1$.

6 Conclusions

This research article proposed a new Painlevé integrable (3+1)-dimensional generalized nonlinear KdV-type equation. The Painlevé analysis confirmed the complete integrability. We constructed the bilinear form in Hirota's D-operators with the Cole-Hopf transformation of the auxiliary function. Using the Hirota bilinear technique, the soliton solutions for the investigated equation were formed in the third order, showing their type as kinks, and the second and third orders have the interactions of kink-solitons. After that, we constructed the rogue wave solutions for the proposed equation utilizing the direct symbolic approach with a logarithmic transformation to construct the bilinear form. We obtained the rogue wave solutions, and in second and third-order solutions, the waves showed the dominating nature of large rogues over the smaller rogues. Using the symbolic software *Mathematica*, we have shown the dynamics for the higher-order solitons and rogue waves with appropriate parameter values. Dynamics for the rogue waves were studied in both transformed and starting variables, which helps to understand the nature of rogues in starting variables concerning the transformed variables.

The proposed equation generalizes the well-known equations having applications in nonlinear sciences and soliton theory. Thus, this equation can explore water wave solutions such as lumps, breathers, and other periodic waves. We studied this equation using two meth-

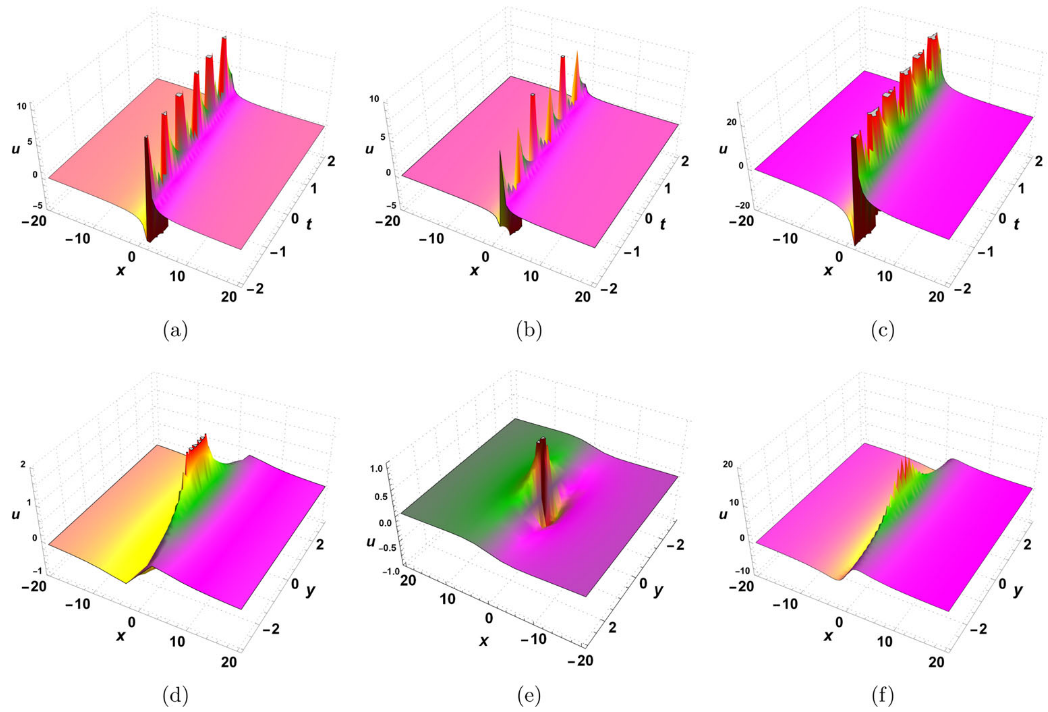

Fig. 5 Dynamical profiles of (33) in the starting variables x, y, z, and t under transformed variables $\xi = x + t$ and $\eta = y + z$. **a**–**c** and **d**–**f** depict 3D profiles in xt and xy-planes, respectively

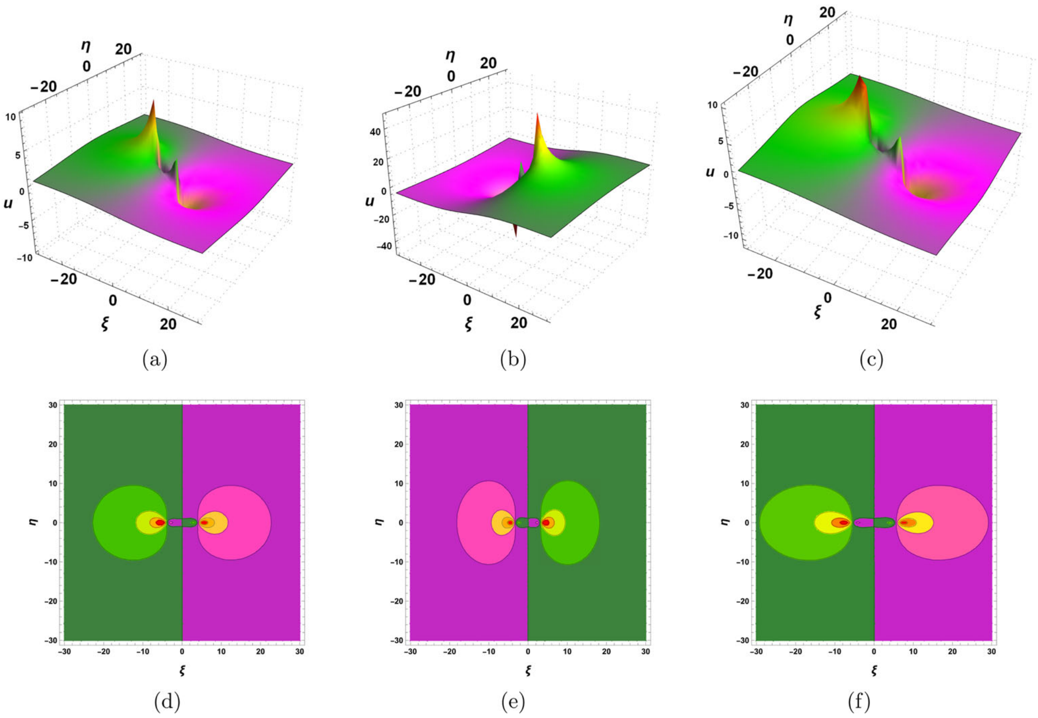

Fig. 6 Dynamics of second-order rogue waves for solution (37) in transformed variables ξ and η. **a**, **b** depicts the 3D profiles in $\xi\eta$-plane, and **d**–**f** shows the contour plots for the **a**–**c**

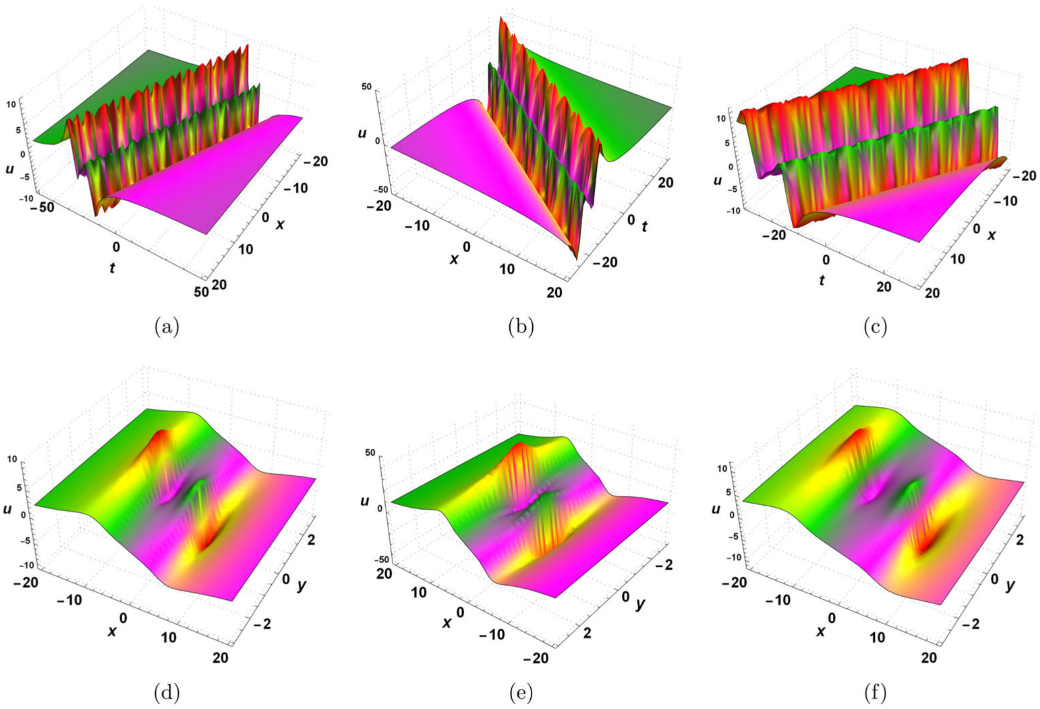

Fig. 7 Dynamical profiles of second-order rogue waves for (37) in the starting variables x, y, z, and t under transformed variables $\xi = x + t$ and $\eta = y + z$. **a**–**c** and **d**–**f** depict 3D profiles in xt and xy-planes, respectively

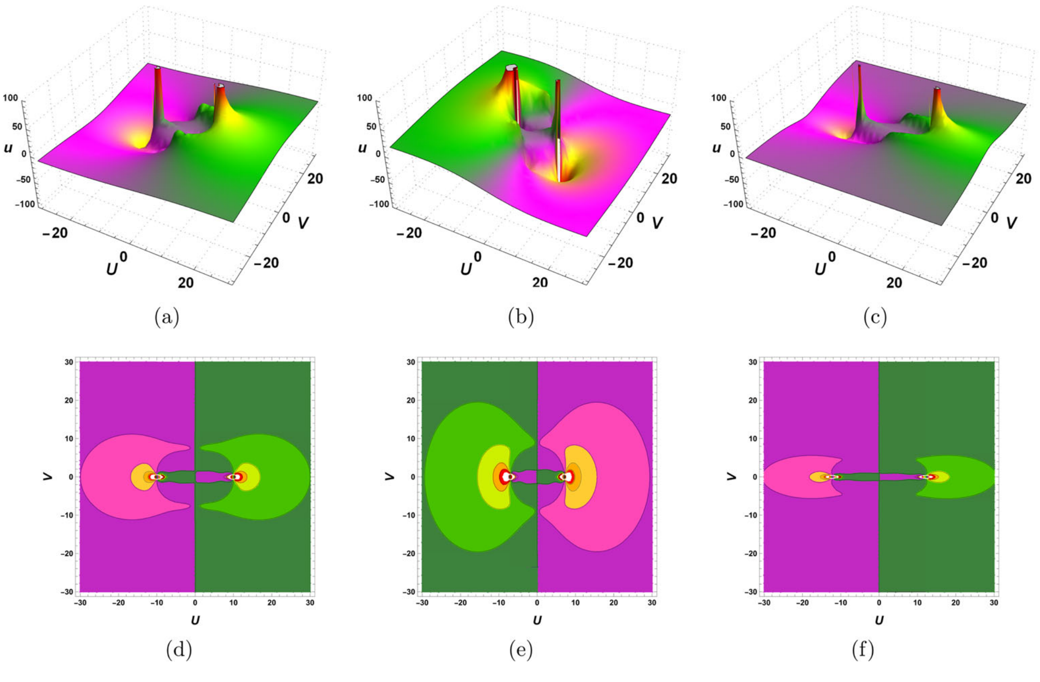

Fig. 8 Dynamics of third-order rogue waves for solution (37) in transformed variables ξ and η. **a**, **b** depicts the 3D profiles in $\xi\eta$-plane, and **d**–**f** shows the contour plots for the **a**–**c**

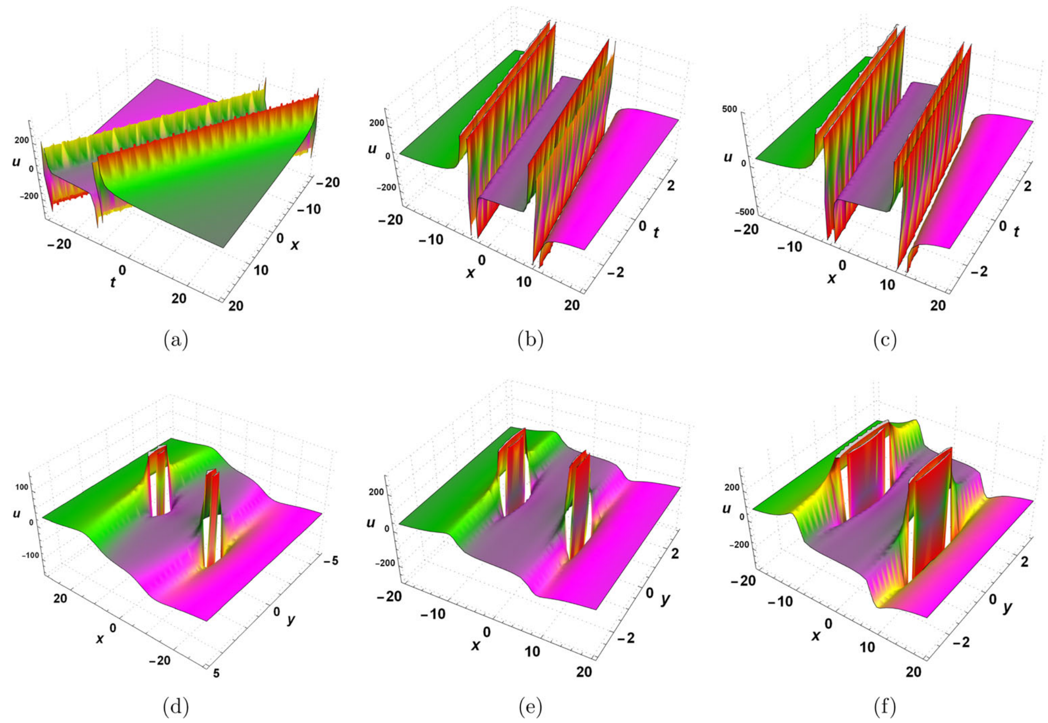

Fig. 9 Dynamical profiles of third-order rogue waves for (41) in the starting variables x, y, z, and t under transformed variables $\xi = x + t$ and $\eta = y + z$. **a**–**c** and **d**–**f** depict 3D profiles in xt and xy-planes, respectively

ods for soliton and rogue wave solutions, so different techniques can be used to construct several other solutions in nonlinear fields. Future work will focus on extending the analysis to other high-dimensional nonlinear systems and exploring potential applications in real-world scenarios such as fluid dynamics, fiber optics, and atmospheric science. This research could provide valuable insights into wave behavior in extreme environments, paving the way for new advances in nonlinear sciences and mathematical physics.

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Declarations

Ethics approval and consent to participate This is not applicable.

Conflict of interest According to the authors, there are no Conflict of interest.

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Research paper

Generalization and analytic exploration of soliton solutions for nonlinear evolution equations via a novel symbolic approach in fluids and nonlinear sciences

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ABSTRACT

In this work, we analyze the new generalized soliton solutions for the nonlinear partial differential equations with a novel symbolic bilinear technique. The proposed approach constructs the soliton solutions depending on the arbitrary parameters, which generalizes the soliton solutions with these additional parameters. Examining phase shifts and their dependence on the parameters influences how solitons collide, merge, or pass through each other, which is essential for the nonlinear analysis of solitons. Using the proposed technique, we examine the well-known (1+1)-dimensional Korteweg–de Vries (KdV) and (2+1)-dimensional Kadomtsev–Petviashvili (KP) equations with a comparative analysis of soliton solutions in the Hirota technique. We construct the generalized solitons solutions for both examined equations up to the third order, providing a better understanding of formed solitons with arbitrary parameter choices. The Cole-Hopf transformations are used to construct the bilinear form in the auxiliary function using Hirota's D-operators for both investigated KdV and KP equations It discusses the phase shift depending on parameters and compares it to the phase shift in Hirota's soliton solutions. We utilize *Mathematica*, a computer algebra system, to obtain the generalized solitons and analyze the dynamic behavior of the obtained solutions by finding the values for the parameters and the relationships among them. Solitons are localized waves that appear in different fields of nonlinear sciences, such as oceanography, plasmas, fluid mechanics, water engineering, optical fibers, and other sciences.

1. Introduction

Investigating solitons in nonlinear fields is a fascinating and pivotal area of research, offering deep insights into the behavior of solitary waves in diverse physical systems. Solitons, as well as stable and localized wave solutions, are essential to study in nonlinear sciences. The nonlinear Korteweg–de Vries (KdV) equation [1], Schrödinger equation [2], and Kadomtsev–Petviashvili (KP) equation [3] are renowned models that apprehend soliton dynamics in various contexts, such as optics, plasma, and fluid dynamics. Understanding solitons' formation, propagation, and interactions provides a deeper understanding of the relation among non-linearity, dispersion, and other relevant factors in nonlinear systems. The study of solitons enhances our theoretical and practical understanding of nonlinear phenomena. It holds practical implications, influencing technological advancements and developing novel applications in ocean engineering, plasma physics, telecommunication, and many nonlinear sciences.

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For decades, Hirota's bilinear method [4,5] and simplified Hirota method [6,7] have been used to obtain the soliton solutions of nonlinear partial differential equations (PDEs) in exact form, which uses the perturbation for the auxiliary function in dependent variable transformation. Now, we have numerous computer algebra systems or symbolic software available, such as ***Maple, Matlab, Mathematica, Octave, Maxima, Scilab***, and some are freeware. Using such software, we can quickly investigate the highly computational work, such as finding exact solutions for nonlinear evolution equations. Researchers use several symbolic or computational methods to study the nonlinear PDE, such as the Hirota's bilinear approach for soliton solutions, symbolic computational approach for rogue wave solutions, simplified Hirota technique for soliton solutions, bilinear Bäcklund transformation for analytic solutions, Painelevé analysis for investigating integrability, and others. Hirota's bilinear method is popular among scientists and researchers for finding the exact soliton solutions, which provides a systematic approach to finding N-soliton solutions where N denotes the number of solitons. We can find several works in the literature to obtain the soliton solution using the Hirota bilinear approach. However, the solutions in Hirota's method do not provide the generalized solutions to study them with several arbitrary parameters.

This work introduces a novel symbolic bilinear technique for constructing generalized soliton solutions for nonlinear partial differential equations. The primary advantage of this method is its ability to incorporate arbitrary parameters into the solution, providing a more flexible and comprehensive framework than traditional methods such as the Hirota bilinear technique. We demonstrate that the proposed approach's solutions are exact and generalized to those obtained through Hirota's bilinear method. The investigation reveals the effectiveness and advantages of the presented approach by comparing the new soliton solutions with those of well-known KdV and KP equations. This flexibility allows for a detailed analysis of the behavior of soliton solutions under varying conditions, which can lead to new insights into the dynamics of these solutions. Moreover, our technique extends Hirota's soliton solutions by adding additional scaling parameters, thus offering a generalized approach that can be applied to a broader range of problems. It is particularly beneficial for examining complex systems where parameter variations can significantly impact the solution's properties. To ensure the robustness and validity of our proposed method, we performed a comparative analysis with soliton solutions obtained using Hirota's technique.

By applying our symbolic bilinear technique to the well-known (1+1)-dimensional Korteweg–de Vries (KdV) [1] and (2+1)-dimensional Kadomtsev–Petviashvili (KP) [3] equations, we verified that the generalized soliton solutions produced are consistent with established solutions, demonstrating the method's accuracy. Moreover, we analyzed the dynamical behavior of these solutions under various parameter values to confirm their stability and physical relevance. Using Mathematica allowed us to systematically explore the parameter space, ensuring that the solutions are mathematically valid and physically meaningful. Additionally, the ability to construct soliton solutions up to the third order highlighted the method's effectiveness, showcasing its potential for handling higher-order and more complex equations. These validations underscore the reliability and efficiency of our novel symbolic bilinear technique. Future studies will strengthen this validation by applying the method to additional nonlinear PDEs and comparing the results with experimental or observational data in relevant physical contexts.

Nonlinear partial differential equations (NLPDEs) represent a vast interdisciplinary domain within physics and applied mathematics. They serve as mathematical models for complex physical systems across diverse scientific fields. Investigating Nonlinear PDEs poses a formidable challenge due to the absence of universal techniques for their analysis. Therefore, each equation necessitates independent examination as a unique problem. However, certain situations may deserve broader approaches. Various techniques are employed to derive analytic and exact solutions, encompassing methodologies such as the Darboux transformation [8–11], simplified Hirota's technique [12,13], Bäcklund transformation [14,15], Bilinear Neural Network Method [16–19], Symbolic computation [20–27], Hirota's bilinear approach [28–31], Symmetry analysis [32,33], Pfaffian technique [34,35], and other methodologies.

The following section proposes the symbolic bilinear technique (SBT) and shows its different steps. In Section 3, we study the (1+1)-D KdV equation utilizing the proposed technique and construct the generalized 1-, 2-, and 3-soliton solutions with their dynamics for distinct parameter values. Section 4 studies the (2+1)-D KP equation using SBT, obtains the generalized 1-, 2-, and 3-soliton solutions for distinct parameters, and shows the dynamics for the constructed solutions. In Section 5, we discuss the findings for the soliton solutions with the proposed technique, and at the end, we conclude the results and the work.

2. Symbolic bilinear technique (SBT)

Symbolic techniques [22–27] for solving nonlinear PDEs offer considerable advantages in mathematical physics and nonlinear sciences. One significant benefit is the ability to derive exact solutions, which provide deep insights into the underlying concepts and serve as benchmarks for validating complex phenomena. These techniques facilitate a deeper analytical understanding of system behavior, revealing relationships between variables and uncovering fundamental properties in soliton theory, plasma physics, and others. Symbolic methods often simplify complex nonlinear PDEs, transforming them into more tractable forms. Techniques like the Cole-Hopf transformation [28,29] and Hirota's bilinear method [30,31] can convert nonlinear equations into linear or bilinear forms, making them easier to solve. Additionally, symbolic techniques offer a systematic approach to finding higher-order solutions, such as solitons and rogue waves, which are crucial for understanding the dynamics of nonlinear systems. Symbolic techniques are versatile and applicable across various scientific disciplines, including fluid dynamics, plasma physics, and optical fibers, making them powerful tools for researchers.

Let us assume a nonlinear partial differential equation of (n+1)-dimensions with n spatial coordinates $\{x_1, x_2, x_3, \ldots, x_n\}$, and one temporal coordinate t as

$$S(u, u_t, u_{x_1}, u_{x_2}, u_{x_3}, u_{x_1x_1}, u_{x_1t}, u_{x_1x_2}, u_{x_1x_1x_1}, \ldots) = 0, \tag{1}$$

which contains partial derivatives with independent variables $\{x_1, x_2, x_3, \ldots, x_n, t\}$ to dependent variable function u.

First we transform Eq. (1) by constructing a Cole-Hopf transformation [28–31]

$$u(x_1, x_2, x_3, \dots, x_n, t) = R(\ln f)_{x_i^m}, \tag{2}$$

where R is a nonzero real constant and $f(x_1, x_2, x_3, \dots, x_n, t)$ is an auxiliary function, m is the order of ith independent variable x_i, obtained by balancing between nonlinear and higher-order terms in PDE for x_i. The Eq. (1) is changed by Cole-Hopf transformation to a bilinear equation in auxiliary function f as

$$T(f, f_t, f_{x_1}, f_{x_2}, f_{x_3}, f_{x_1x_1}, f_{x_1t}, f_{x_1x_2}, f_{x_1x_1x_1}, \dots) = 0, \tag{3}$$

which can be represented in the Hirota's bilinear form with D-operators as

$$H(D_t, D_{x_1}, D_{x_2}, D_{x_3}, D_{x_1}^2, D_{x_1}D_{x_2}, D_{x_2}^2, D_{x_2}D_{x_3}, \dots)f.f = 0. \tag{4}$$

For obtaining the N-soliton solution for Eq. (4), we express the auxiliary function f symbolically as

$$f = \sum_{k_{\{i=1,2,\dots,N\}}=0,1}^{\{2^N\}} a_{k_1,k_2,k_3,\dots,k_N} e^{k_1\xi_1+k_2\xi_2+k_3\xi_3+\dots+k_N\xi_N}, \tag{5}$$

where $k_i = 0, 1$ are the binary choices for $1 \le i \le N$, 2^N represents the number of terms, $a_{k_1,k_2,k_3,\dots,k_N}$ are the real non-zero parameters to be determined, and ξ_i are the phases for Eq. (4)

For $N = 1$, we get $k_1 = 0, 1$, therefore

$$f = \sum_{k_1=0,1}^{\{2\}} a_{k_1} e^{k_1\xi_1} = a_0 + a_1 e^{\xi_1}.$$

For $N = 2$, we have $k_1, k_2 = 0, 1$, therefore

$$f = \sum_{k_1,k_2=0,1}^{\{4\}} a_{k_1,k_2} e^{k_1\xi_1+k_2\xi_2} = a_{0,0} + a_{1,0}e^{\xi_1} + a_{0,1}e^{\xi_2} + a_{1,1}e^{\xi_1+\xi_2}.$$

For $N = 3$, we get $k_1, k_2, k_3 = 0, 1$ therefore

$$f = \sum_{k_1,k_2,k_3=0,1}^{\{8\}} a_{k_1,k_2,k_3} e^{k_1\xi_1+k_2\xi_2+k_3\xi_3}$$
$$= a_{0,0,0} + a_{1,0,0}e^{\xi_1} + a_{0,1,0}e^{\xi_2} + a_{0,0,1}e^{\xi_3} + a_{1,1,0}e^{\xi_1+\xi_2} + a_{1,0,1}e^{\xi_1+\xi_3} + a_{0,1,1}e^{\xi_2+\xi_3} + a_{1,1,1}e^{\xi_1+\xi_2+\xi_3}.$$

Thus, the auxiliary functions f provide the N-soliton solutions of the bilinear Eq. (4) for different values of $N = 1, 2, 3$, that are solutions to Eq. (1). The symbolic bilinear technique obtains the generalized N-soliton solutions depending on arbitrary parameters and observes that the Hirota's N-soliton solutions [1] using bilinear method [29–31] are as one case for the obtained solution by this symbolic approach.

For $a_0 = a_1 = 1$, $a_{0,0} = a_{1,0} = a_{0,1} = 1$, and $a_{0,0,0} = a_{1,0,0} = a_{0,1,0} = a_{0,0,1} = 1$, the above auxiliary functions generate Hirota's [1] 1-soliton, 2-soliton, and 3-soliton solutions, respectively. Thus, the solutions by this approach have the opportunities to observe and study the behavior of solitons with different values of these additional real parameters $a_{k_1,k_2,k_3,\dots,k_N}$ along with the constants presents in the phase variables for the studied equations as discussed in the following section.

3. (1+1)-dimensional KdV equation

The nonlinear KdV equation [1] is an evolution equation that describes the evolution of one-dimensional, weakly nonlinear, and long waves. It was first introduced in the field of hydrodynamics to model the behavior of shallow water waves. The KdV equation is

$$u_t + 6uu_x + u_{xxx} = 0, \tag{6}$$

where u is the dependent variable that represents the wave amplitude, x and t are the spatial coordinate and time, respectively. The Eq. (6) is particularly notable for its soliton solutions, which are solitary wave solutions that maintain their shape and speed during propagation. These solitons arise due to a balance among nonlinear and dispersive terms in the equation. The nonlinear KdV equation has applications in several domains, including plasma physics, fluid dynamics, and nonlinear optics, making it a fundamental model for studying wave phenomena.

Let us consider the phase variable ξ_i in the KdV Eq. (6) as

$$\xi_i = \mu_i x - d_i t, \tag{7}$$

with $\mu_i : i = 1, 2, \dots,$ as constant parameters and d_i as the dispersion coefficients. On putting $u = e^{\xi_i}$ in linear terms of Eq. (6), we get the dispersion $d_i = \mu_i^3$.

Considering the transformation

$$u(x, t) = R(\ln f)_{xx}, \tag{8}$$

and putting it with $f(x,t) = 1 + e^{\xi_1}$ in Eq. (6). On solving, we get R as 2. Thus, Eq. (8) transforms Eq. (6) into a bilinear equation in f as

$$f f_{xt} - f_x f_t + 3f_{xx}^2 - 4f_x f_{xxx} + f f_{xxxx} = 0. \tag{9}$$

Using the bilinear differential operators $D_i : i = X, T$ defined by Hirota [1] as

$$D_X^{R_1} D_T^{R_2} UV = \left(\frac{\partial}{\partial_X} - \frac{\partial}{\partial_{X'}}\right)^{R_1} \left(\frac{\partial}{\partial_T} - \frac{\partial}{\partial_{T'}}\right)^{R_2} U(X,T)V(X',T')|_{X=X',T=T'},$$

with X' and T' as the formal variables and $R_j : j = 1, 2$ as positive integers. Thus, the bilinear Eq. (9) can be written in the Hirota's bilinear form with bilinear differentials D as

$$[D_x D_t + D_x^4] f.f = 0. \tag{10}$$

For one soliton solution, we take the function f in Eq. (10) by having $N = 1$ in Eq. (5) as

$$f_1 = f(x,t) = \sum_{k_1=0,1}^{\{2\}} a_{k_1} e^{k_1 \xi_1} = a_0 + a_1 e^{\xi_1} = a_0 + a_1 e^{\mu_1 x - d_1 t}, \tag{11}$$

which satisfies the bilinear Eq. (10) identically with arbitrary choices of a_0 and a_1. By substituting the Eqs. (11) into (8), we get a generalized solution for one-soliton as

$$u_1(x,t) = u = \frac{2a_0 a_1 \mu_1^2 e^{\mu_1^3 t + \mu_1 x}}{\left(a_0 e^{\mu_1^3 t} + a_1 e^{\mu_1 x}\right)^2}, \tag{12}$$

that dependents on the arbitrary parameters a_0 and a_1. Therefore, we analyze the behavior and dynamics of this 1-soliton solution for distinct values of these parameters. As, we have discussed that for $a_0 = a_1 = 1$ the solution will give the Hirota's 1-soliton solution [1], thus, the obtained solution is a generalized 1-soliton solution with arbitrary choice of non-zero parameters, and the dynamics are shown in Fig. 1.

With $N = 2$, Eq. (5) gives auxiliary function f as

$$f_2 = f = \sum_{k_1,k_2=0,1}^{\{4\}} a_{k_1,k_2} e^{k_1\xi_1 + k_2\xi_2} = a_{0,0} + a_{1,0} e^{\xi_1} + a_{0,1} e^{\xi_2} + a_{1,1} e^{\xi_1+\xi_2}. \tag{13}$$

By substituting the Eq. (13) into bilinear Eq. (10), and equating the coefficients of distinct expressions in power of exponential functions to zero, we get

$$a_{1,1} = \frac{(\mu_1 - \mu_2)^2 a_{0,1} a_{1,0}}{(\mu_1 + \mu_2)^2 a_{0,0}}. \tag{14}$$

On substituting Eqs (13) into (8), give a two-soliton solution of Eq. (6) as

$$u_2(x,t) = u = 2(\ln f_2)_{xx}, \tag{15}$$

that dependents on the arbitrary parameters $a_{0,0}, a_{0,1}$ and $a_{1,0}$. Therefore, we study the dynamical behavior of this 2-soliton solution for distinct values of these arbitrary non-zero parameters. For $a_{0,0} = a_{0,1} = a_{1,0} = 1$ the solution (15) will represent a Hirota's 2-soliton solution [1] and Eq. (14) shows the phase shift in Hirota's bilinear method. Thus, the obtained solution is a generalized 2-soliton solution with these parameters, and Fig. 2 shows the dynamics for this solution.

For $N = 3$ in Eq. (5), we consider the auxiliary function f as

$$\begin{aligned} f_3 = f &= \sum_{k_1,k_2,k_3=0,1}^{\{8\}} a_{k_1,k_2,k_3} e^{k_1\xi_1 + k_2\xi_2 + k_3\xi_3} \\ &= a_{0,0,0} + a_{1,0,0} e^{\xi_1} + a_{0,1,0} e^{\xi_2} + a_{0,0,1} e^{\xi_3} + a_{1,1,0} e^{\xi_1+\xi_2} + a_{1,0,1} e^{\xi_1+\xi_3} + a_{0,1,1} e^{\xi_2+\xi_3} + a_{1,1,1} e^{\xi_1+\xi_2+\xi_3}. \end{aligned} \tag{16}$$

On substituting the Eq. (16) into the bilinear Eq. (10), and equating the coefficients of distinct expressions in power of exponential functions to zero, we get

$$a_{1,1,0} = \frac{(\mu_1 - \mu_2)^2 a_{0,1,0} a_{1,0,0}}{(\mu_1 + \mu_2)^2 a_{0,0,0}}, \qquad a_{0,1,1} = \frac{(\mu_2 - \mu_3)^2 a_{0,0,1} a_{0,1,0}}{(\mu_2 + \mu_3)^2 a_{0,0,0}}, \qquad a_{1,0,1} = \frac{(\mu_1 - \mu_3)^2 a_{0,0,1} a_{1,0,0}}{(\mu_1 + \mu_3)^2 a_{0,0,0}},$$

$$a_{1,1,1} = \frac{(\mu_1 - \mu_2)^2 (\mu_1 - \mu_3)^2 (\mu_2 - \mu_3)^2 a_{0,0,1} a_{0,1,0} a_{1,0,0}}{(\mu_1 + \mu_2)^2 (\mu_1 + \mu_3)^2 (\mu_2 + \mu_3)^2 a_{0,0,0}^2},$$

which shows the relation as

$$a_{1,1,1} = (a_{1,1,0} \times a_{0,1,1} \times a_{1,0,1}) \frac{a_{0,0,0}}{a_{1,0,0} a_{0,1,0} a_{0,0,1}}, \tag{17}$$

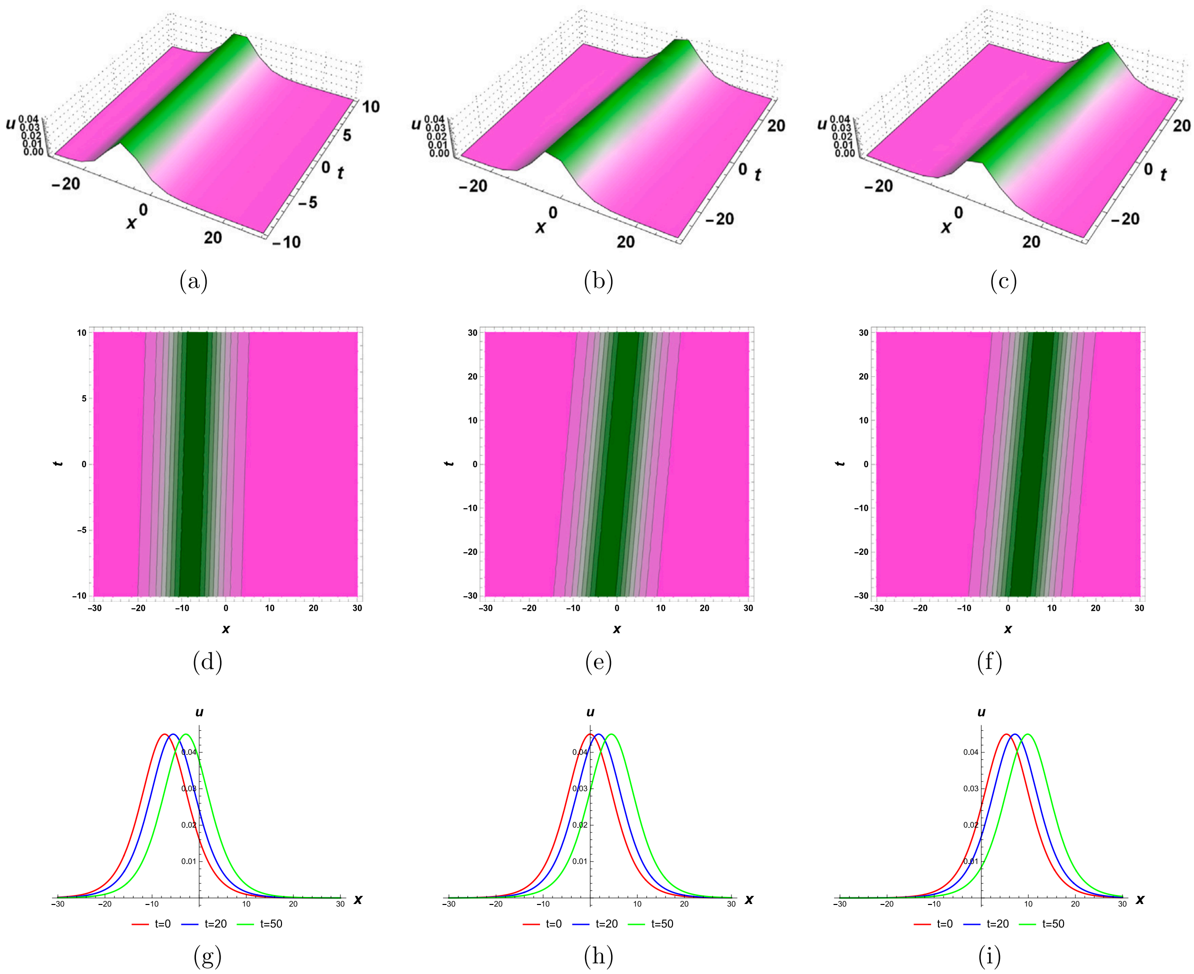

Fig. 1. Dynamical profiles of 1-soliton solitons for (12) with $\mu_1 = 0.3$ and **(a)** $a_0 = 0.1, a_1 = 0.9$; **(b)** $a_0 = 1, a_1 = 1$ **(c)** $a_0 = -0.5, a_1 = -0.1$; **(d)–(f)** and **(g)–(i)** depicts the contour and 2D plots for **(a)–(c)**, respectively.

where $a_{0,0,0}, a_{1,0,0}, a_{0,1,0}$ and $a_{0,0,1}$ are arbitrary constants. Thus Eq. (16) satisfies Eq. (10) as a solution with the above parameters. By substituting Eq. (16) into (8), we establish a 3-soliton solution as

$$u_3(x,t) = u = 2(\ln f_3)_{xx}, \tag{18}$$

that dependents on the arbitrary parameters $a_{0,0,0}, a_{1,0,0}, a_{0,1,0}$ and $a_{0,0,1}$. Therefore, we study the behavior with dynamics of this 3-soliton solution for distinct values of these non-zero parameters. For $a_{0,0,0} = a_{1,0,0} = a_{0,1,0} = a_{0,0,1} = 1$ the solution (18) will represent a Hirota's 3-soliton solution [1] and Eq. (17) satisfies the dispersion relation for the parameters as in Hirota's bilinear method, thus, the obtained solution is a generalized three-soliton solution with arbitrary parameters, and the dynamics are shown in Fig. 3.

4. (2+1)-dimensional KP equation

The nonlinear KP equation [3] significantly extends the Korteweg–de Vries (KdV) equation, specifically developed to describe two-dimensional weakly nonlinear and dispersive waves. Its mathematical form is

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} = 0, \tag{19}$$

where u represents the wave amplitude, x, y and t are spatial and time coordinate, respectively. The nonlinear KP equation is essential to studying solitons and integrable systems exhibiting rich and complex behaviors of wave interactions. Similar to the KdV equation, the KP equation supports solitons, but its two-dimensional nature allows for more elaborate structures, such as two-soliton solutions that interact in a nontrivial manner. The nonlinear KP equation finds applications in diverse fields, such as plasma, fluid mechanics, and oceanography, contributing to our understanding of nonlinear wave phenomena in multiple dimensions.

Let us consider the phase variable ξ_i in the KP Eq. (19) as

$$\xi_i = \mu_i x + \nu_i y - d_i t, \tag{20}$$

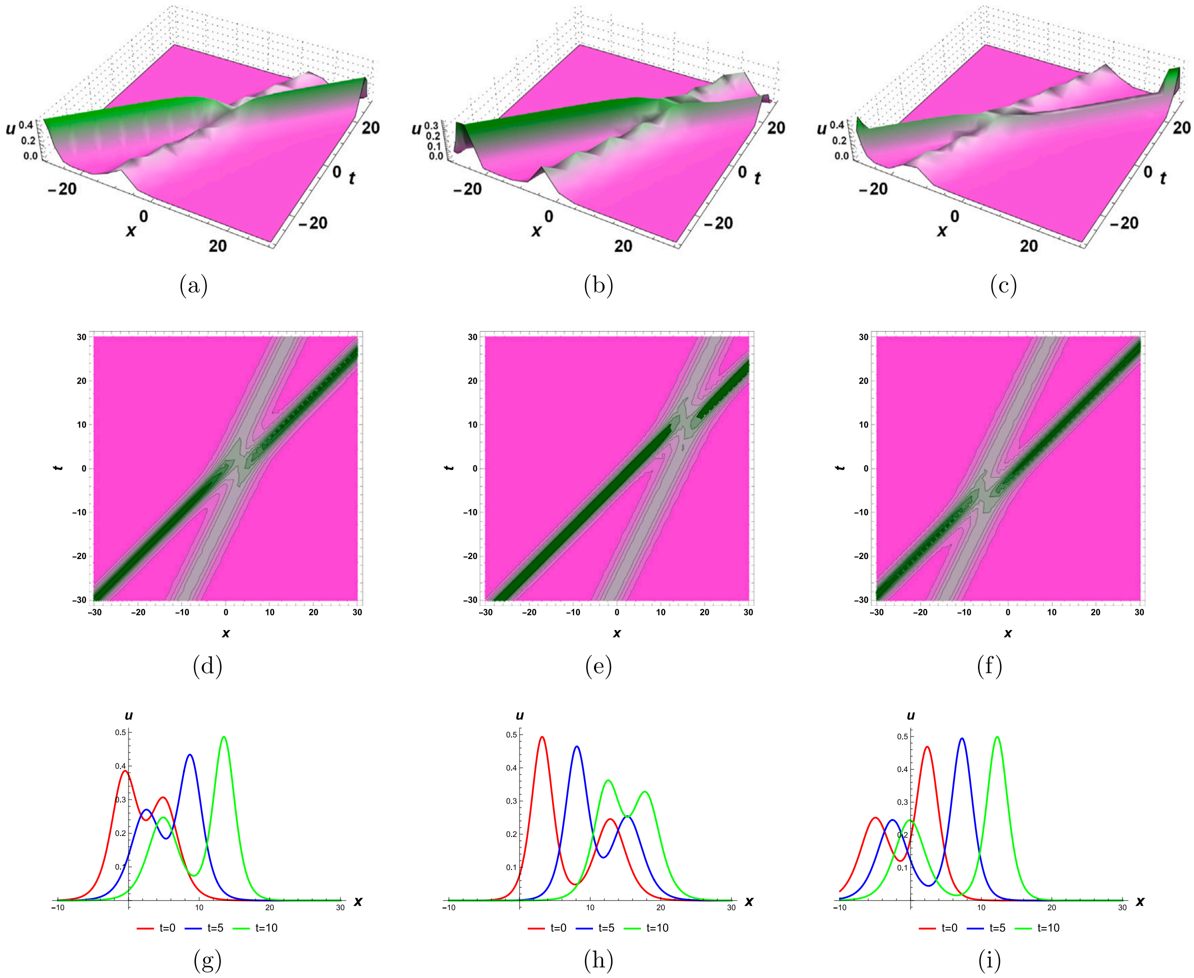

Fig. 2. Dynamical profiles of 2-soliton solitons for (15) with $\mu_1 = 0.7, \mu_2 = 1$ and **(a)** $a_{0,0} = 1, a_{1,0} = 1, a_{0,1} = 1$ **(b)** $a_{0,0} = 25, a_{1,0} = 0.5, a_{0,1} = 1$ **(c)** $a_{0,0} = 0.3, a_{1,0} = 10, a_{0,1} = 1$; **(d)–(f)** and **(g)–(i)** depicts the contour and 2D plots for **(a)–(c)**, respectively.

with $\mu_i, \nu_i : i = 1, 2, \ldots,$ as constant parameters and d_i as the dispersion coefficients. On putting $u = e^{\xi_i}$ in linear terms of Eq. (19), we get the dispersion $d_i = \frac{\mu_i^4 - \nu_i^2}{\mu_i}$.

Considering the transformation

$$u(x, y, t) = R(\ln f)_{xx}, \tag{21}$$

and putting it with $f(x, y, t) = 1 + e^{\xi_1}$ in Eq. (19). On solving, we get R as 2. Thus, Eq. (21) transforms Eq. (19) into a bilinear equation in f as

$$f f_{xt} - f_x f_t + 3f_{xx}^2 - 4f_x f_{xxx} + f f_{xxxx} - f f_{yy} + f_y^2 = 0. \tag{22}$$

Using the bilinear operators $D_i : i = X, Y, T$ defined by Hirota [1] as

$$D_X^{R_1} D_Y^{R_2} D_T^{R_3} UV = \left(\frac{\partial}{\partial X} - \frac{\partial}{\partial X'}\right)^{R_1} \left(\frac{\partial}{\partial Y} - \frac{\partial}{\partial Y'}\right)^{R_2} \left(\frac{\partial}{\partial T} - \frac{\partial}{\partial T'}\right)^{R_3} U(X, Y, T)V(X', Y', T')|_{X=X', Y=Y', T=T'},$$

with X', Y' and T' as the formal variables and $R_j : j = 1, 2, 3$ as positive integers. Thus, the bilinear Eq. (22) can be written in the Hirota's bilinear form with bilinear differentials D as

$$[D_x D_t + D_x^4 - D_y^2] f.f = 0. \tag{23}$$

For $N = 1$ in Eq. (5), we take the function f in Eq. (23) as

$$f_1 = f(x, y, t) = \sum_{k_1=0,1}^{\{2\}} a_{k_1} e^{k_1 \xi_1} = a_0 + a_1 e^{\xi_1} = a_0 + a_1 e^{\mu_1 x + \nu_1 y - d_1 t}, \tag{24}$$

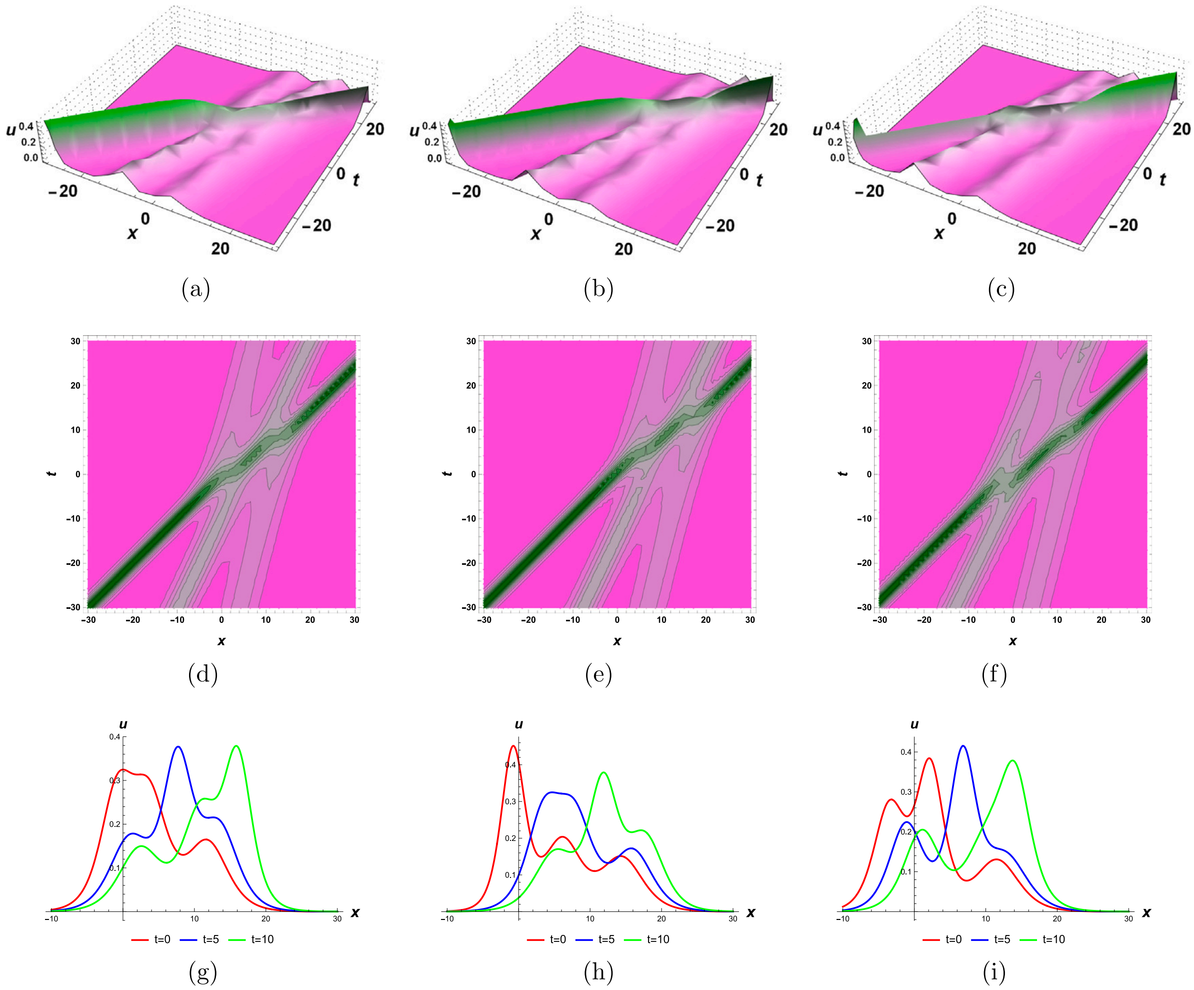

Fig. 3. Dynamical profiles of 3-soliton solitons for (15) with $\mu_1 = 0.7, \mu_2 = 1, \mu_3 = 0.5$ and **(a)** $a_{0,0,0} = a_{1,0,0} = a_{0,1,0} = a_{0,0,1} = 1$ **(b)** $a_{0,0,0} = 4, a_{1,0,0} = 1, a_{0,1,0} = 8, a_{0,0,1} = 1$ **(c)** $a_{0,0,0} = 1, a_{1,0,0} = 10, a_{0,1,0} = 5, a_{0,0,1} = 1$; **(d)–(f)** and **(g)–(i)** depicts the contour and 2D plots for **(a)–(c)**, respectively.

which satisfies the bilinear Eq. (23) identically with arbitrary choices of a_0 and a_1. By substituting the Eqs. (24) into (21), we get one-soliton solution as

$$u_1(x,y,t) = u = \frac{2a_0a_1\mu_1^2 e^{\frac{\nu_1^2 t}{\mu_1}+\mu_1^3 t+\mu_1 x+\nu_1 y}}{\left(a_1 e^{\frac{\nu_1^2 t}{\mu_1}+\mu_1 x+\nu_1 y}+a_0 e^{\mu_1^3 t}\right)^2}, \tag{25}$$

that dependents on the parameters a_0 and a_1. Therefore, we observe the behavior and the dynamics of this 1-soliton solution for distinct values of these non-zero parameters. For $a_0 = a_1 = 1$ the solution will represent the Hirota's 1-soliton solution [3], thus, the obtained solution is a generalized 1-soliton solutions with these non-zero parameters, and the dynamics are shown in Fig. 4.

Having $N = 2$ in Eq. (5), we take auxiliary function f as

$$f_2 = f = \sum_{k_1,k_2=0,1}^{\{4\}} a_{k_1,k_2} e^{k_1\xi_1+k_2\xi_2} = a_{0,0} + a_{1,0}e^{\xi_1} + a_{0,1}e^{\xi_2} + a_{1,1}e^{\xi_1+\xi_2}. \tag{26}$$

By substituting the Eq. (26) into the bilinear Eq. (23), and equating the coefficients of distinct expressions in power of exponential functions to zero, we get

$$a_{1,1} = \frac{\left(3\mu_1^2\mu_2^2(\mu_1-\mu_2)^2 + (\mu_1\nu_2-\mu_2\nu_1)^2\right)a_{0,1}a_{1,0}}{\left(3\mu_1^2\mu_2^2(\mu_1+\mu_2)^2 + (\mu_1\nu_2-\mu_2\nu_1)^2\right)a_{0,0}}. \tag{27}$$

On substituting Eq. (26) into Eq. (21), give a two-soliton solution of Eq. (19) as

$$u_2(x,t) = u = 2(\ln f_2)_{xx}, \tag{28}$$

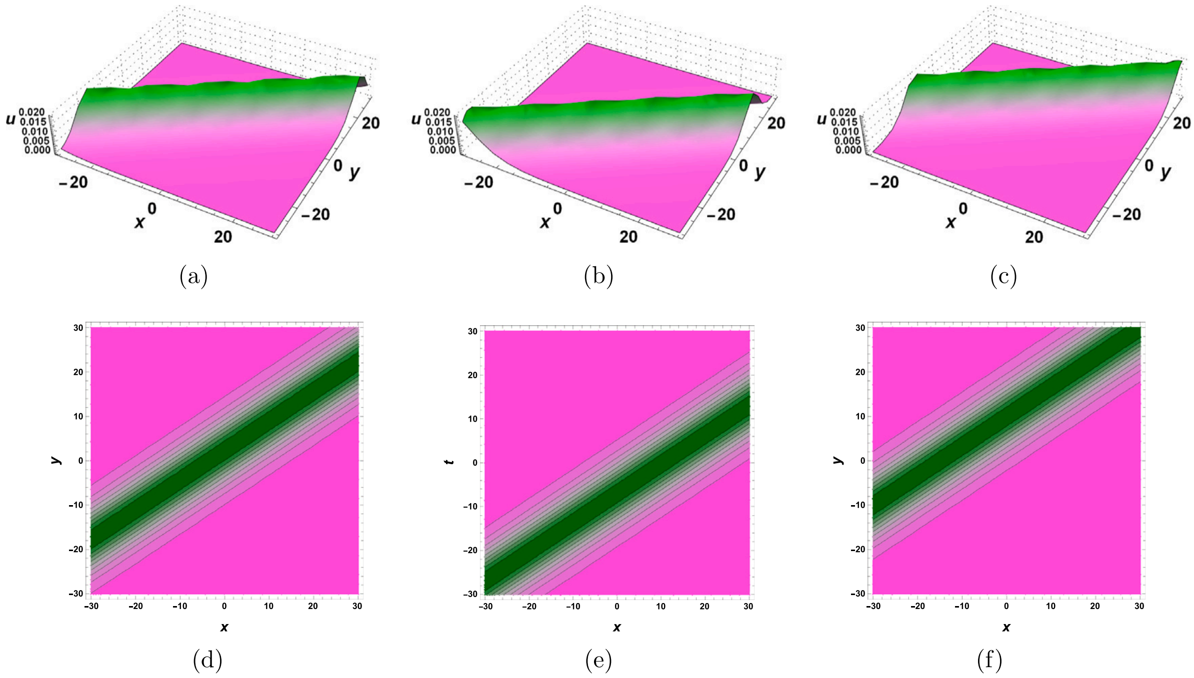

Fig. 4. Dynamical profiles of 1-soliton solutions for (25) with $\mu_1 = 0.2, \nu_1 = -0.3, t = 0$ and **(a)** $a_0 = 0.1, a_1 = 0.2$; **(b)** $a_0 = 0.8, a_1 = 0.11$ **(c)** $a_0 = 1, a_1 = 20$; **(d)–(f)** depicts the contour plots for **(a)–(c)**, respectively.

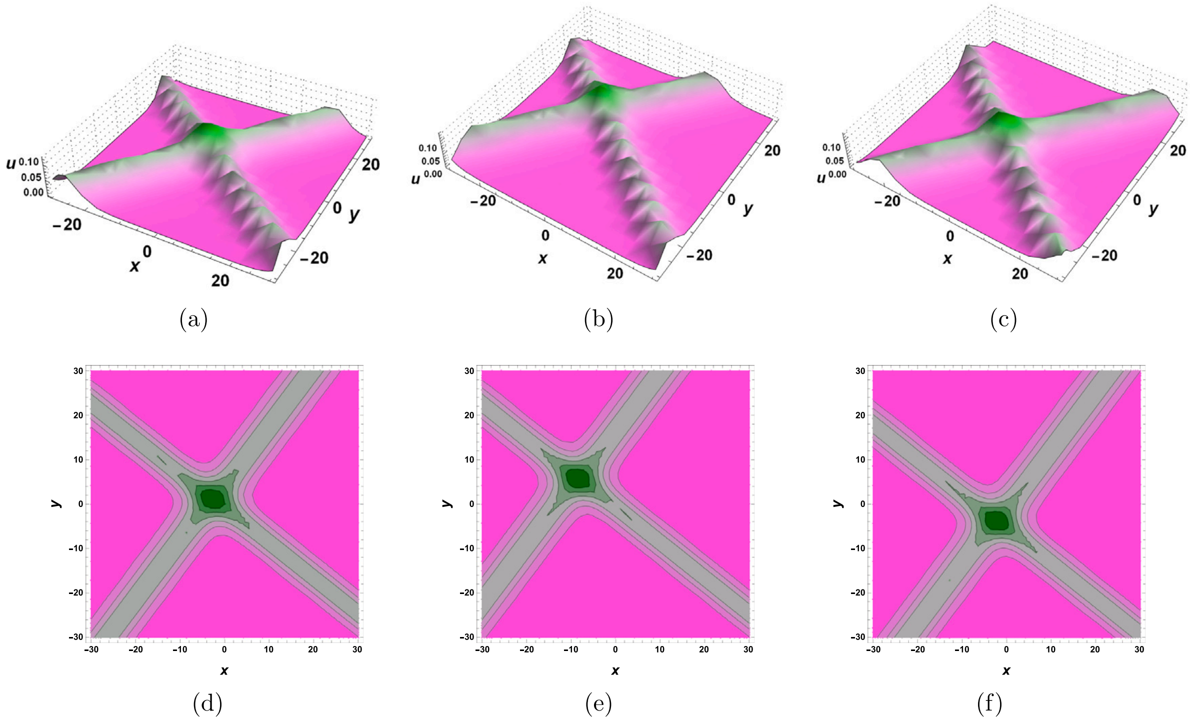

Fig. 5. Dynamical profiles of 2-soliton solutions for (15) with $\mu_1 = \mu_2 = 0.4, \nu_1 = 0.5, \nu_2 = -0.3, t = 0$ and **(a)** $a_{0,0} = 0.1, a_{1,0} = 0.2, a_{0,1} = 0.5$ **(b)** $a_{0,0} = 0.1, a_{1,0} = 0.2, a_{0,1} = 20$ **(c)** $a_{0,0} = 1, a_{1,0} = 20, a_{0,1} = 1$; **(d)–(f)** depicts the contour plots for **(a)–(c)**, respectively.

that dependents on the arbitrary parameters $a_{0,0}, a_{0,1}$ and $a_{1,0}$. Therefore, we study the dynamics of this 2-soliton solution for distinct values of these non-zero parameters. For $a_{0,0} = a_{0,1} = a_{1,0} = 1$, the solution (28) will represent a Hirota's 2-soliton solution [3] and Eq. (27) shows the relation for the phase shift as in Hirota's bilinear method, thus, the obtained solution is a generalized 2-soliton solution with these arbitrary parameters, and Fig. 5 shows the dynamics of the solution.

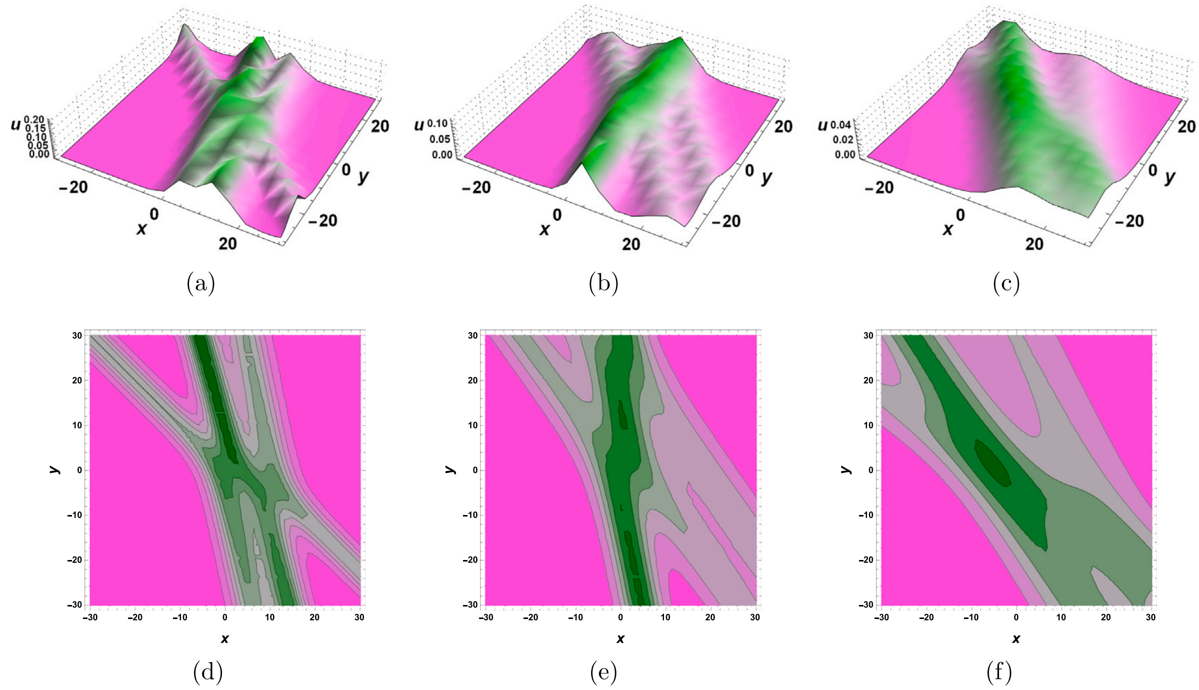

Fig. 6. Dynamical profiles of 3-soliton solutions for (31) with $t = 0$ and **(a)** $a_{0,0,0} = a_{1,0,0} = a_{0,1,0} = a_{0,0,1} = 1, \mu_1 = \mu_3 = 0.5, \mu_2 = 0.7, \nu_1 = 0.5, \nu_2 = 0.2, \nu_3 = 0.1$ **(b)** $a_{0,0,0} = 0.5, a_{1,0,0} = 0.3, a_{0,1,0} = 0.5, a_{0,0,1} = 1, \mu_1 = 0.4, \mu_2 = 0.3, \mu_3 = 0.5, \nu_1 = 0.3, \nu_2 = 0.2, \nu_3 = 0.1$ **(c)** $a_{0,0,0} = 0.1, a_{1,0,0} = 0.3, a_{0,1,0} = 0.5, a_{0,0,1} = 0.1, \mu_1 = 0.2, \mu_2 = 0.3, \mu_3 = 0.2, \nu_1 = 0.3, \nu_2 = 0.2, \nu_3 = 0.1$; **(d)–(f)** depicts the contour plots for **(a)–(c)**, respectively.

For $N = 3$ in Eq. (5), we consider the function f as

$$f_3 = f = \sum_{k_1,k_2,k_3=0,1}^{\{8\}} a_{k_1,k_2,k_3} e^{k_1\xi_1+k_2\xi_2+k_3\xi_3} = a_{0,0,0} + a_{1,0,0}e^{\xi_1} + a_{0,1,0}e^{\xi_2} + a_{0,0,1}e^{\xi_3} + a_{1,1,0}e^{\xi_1+\xi_2} + a_{1,0,1}e^{\xi_1+\xi_3} + a_{0,1,1}e^{\xi_2+\xi_3} + a_{1,1,1}e^{\xi_1+\xi_2+\xi_3}. \tag{29}$$

On substituting Eq. (29) into the bilinear Eq. (23), and equating the coefficients of distinct expressions in power of exponential functions to zero, we get

$$\begin{aligned}
a_{1,1,0} &= \frac{\left(3\mu_1^2\mu_2^2(\mu_1-\mu_2)^2 + (\mu_1\nu_2-\mu_2\nu_1)^2\right) a_{1,0,0}a_{0,1,0}}{\left(3\mu_1^2\mu_2^2(\mu_1+\mu_2)^2 + (\mu_1\nu_2-\mu_2\nu_1)^2\right) a_{0,0,0}},\\
a_{0,1,1} &= \frac{\left(3\mu_2^2\mu_3^2(\mu_2-\mu_3)^2 + (\mu_2\nu_3-\mu_3\nu_2)^2\right) a_{0,1,0}a_{0,0,1}}{\left(3\mu_2^2\mu_3^2(\mu_2+\mu_3)^2 + (\mu_2\nu_3-\mu_3\nu_2)^2\right) a_{0,0,0}},\\
a_{1,0,1} &= \frac{\left(3\mu_1^2\mu_3^2(\mu_1-\mu_3)^2 + (\mu_1\nu_3-\mu_3\nu_1)^2\right) a_{1,0,0}a_{0,0,1}}{\left(3\mu_1^2\mu_3^2(\mu_1+\mu_3)^2 + (\mu_1\nu_3-\mu_3\nu_1)^2\right) a_{0,0,0}},\\
a_{1,1,1} &= (a_{1,1,0}\times a_{0,1,1}\times a_{1,0,1})\frac{a_{0,0,0}}{a_{1,0,0}a_{0,1,0}a_{0,0,1}},
\end{aligned} \tag{30}$$

where $a_{0,0,0}, a_{1,0,0}, a_{0,1,0}$ and $a_{0,0,1}$ are arbitrary constants. Thus Eq. (29) satisfies Eq. (23) as a solution with the arbitrary non-zero parameters. By substituting Eq. (29) into (21), we establish a 3-soliton solution as

$$u_3(x, y, t) = u = 2(\ln f_3)_{xx}, \tag{31}$$

that depends on the arbitrary parameters $a_{0,0,0}, a_{1,0,0}, a_{0,1,0}$ and $a_{0,0,1}$. Therefore, we study the dynamics of this 3-soliton solution for distinct values of these non-zero parameters. For $a_{0,0,0} = a_{1,0,0} = a_{0,1,0} = a_{0,0,1} = 1$ the solution will represent Hirota's 3-soliton solution [3] and Eq. (30) satisfies the dispersion relation for the parameters as in Hirota's bilinear technique, thus, the obtained solution is a generalized three-soliton solution with these non-zero parameters, and the dynamics are shown in Fig. 6.

5. Results and discussion

This work has analyzed the newly constructed generalized soliton solutions concerning arbitrary parameters utilizing the proposed symbolic bilinear technique. Our analysis includes the examination of phase shifts and their dependence on the parameters, which is essential for characterizing the interaction of solitons in physical systems [1–4]. Additionally, the generalized solitons produced by our approach provide a more realistic representation of physical processes due to the inclusion of arbitrary parameters. This approach verifies that the solutions obtained in Hirota's bilinear approach [1,3] for the studied equations are a case for the established generalized solutions. We have shown a comparative analysis of the existing solutions for well-known KdV and KP equations using Hirota's bilinear method and the solutions using our proposed approach. Researchers and investigators can apply this technique to the other equations to more broadly understand the behavior and physical appearance of the solutions for a nonlinear system [5–9].

The analysis of generalized soliton solutions using the symbolic bilinear technique reveals several critical aspects of their physical significance. These include the flexibility introduced by arbitrary parameters, the importance of phase shifts in soliton interactions, and the validation of the proposed approach against established methods. Generalized soliton solutions that arbitrarily provide additional parameters make the description of physical processes more universal and complete, giving a better understanding of their underlying dynamics. The study of phase shifts as a function of these parameters is particularly relevant for understanding soliton interactions in physical systems. Otherwise, phase shifts can play a crucial role in influencing the collisions of solitons to merge like this or cross each other and are, therefore, essential for the nonlinear study of interaction type. It is also checked by the obtained solution being exact and generalizable concerning Hirota's bilinear method, confirming its reliability for a new technique. This study also compares the new solutions with well-known KdV and KP equations to show that the proposed method is effective against known results and satisfactory compared to available approaches. Here, this comparison illustrates the enhancements and deviations inserted via arbitrary parameters to give a new understanding of soliton behavior. It is essential for applications where soliton behavior plays a key role, as in our case, one elementary problem can lead to N-solitons. In addition, its generality for other nonlinear equations makes it worthwhile in the large toolbox of techniques used by researchers who explore various phenomena within nonlinear dynamics.

The physical significance lies in the enhanced realism and versatility of soliton solutions with arbitrary parameters, improved understanding of soliton interactions through phase shift analysis, validation against established approaches, and the potential for broad application in studying nonlinear systems. Selecting several arbitrary parameters, we have generated N-soliton up to $N = 3$ with the given symbolic bilinear technique and analyzed the structures for the obtained solutions dynamically. We explains the analysis as follows:

- Figs. 1 and 4 plot the one solitons in (a) to (c) for investigated KdV and KP equations, and analyses the soliton behavior for different values of arbitrary parameters a_0 and a_1 with the constant $\mu_1 = 0.3$ for KdV equation and $\mu_1 = 0.2, \nu_1 = -0.3, t = 0$ for KP equation. The solitons change their position with respect to the singularities depending on the parameters a_0 and a_1. Graphics (d) to (f) show the contour plots for (a) to (c), respectively. The 2D graphics (g) to (h) in Fig. 1 depict that the solitons are moving in right direction of x-axis.
- In Figs. 2 and 5, we illustrate the interactions of two solitons in (a) to (c) for investigated KdV and KP equations, and analyses the solitons behavior for different values of arbitrary parameters $a_{0,0}, a_{0,1}$ and $a_{1,0}$ with the constants $\mu_1 = 0.7, \mu_2 = 1$ for the KdV equation and $\mu_1 = \mu_2 = 0.4, \nu_1 = 0.5, \nu_2 = -0.3, t = 0$ for the KP equation. The solitons change their interaction position with respect to the singularities depending on these parameters. Graphics (d) to (f) show the contour plots for (a) to (c), respectively. The 2D graphics (g) to (h) in Fig. 2 depict that the solitons interactions moving in right direction of x-axis.
- Figs. 3 and 6 show the interactions of three solitons in (a) to (c) for investigated KdV and KP equations, and analyze the solitons behavior for different values of arbitrary parameters $a_{0,0,0}, a_{0,0,1}, a_{0,1,0}$ and $a_{1,0,0}$ with the constants $\mu_1 = 0.7, \mu_2 = 1, \mu_3 = 0.5$ for the KdV equation and different values of constants $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3, t = 0$ for the KP equation. The soliton interactions change their position with respect to the singularities depending on these parameters. Graphics (d) to (f) show the contour plots for (a) to (c), respectively. The 2D graphics (g) to (h) in Fig. 3 depict that the solitons interactions moving in right direction of x-axis.

6. Conclusions

This research study analyzed the newly constructed generalized soliton solution for the well-known KdV and KP nonlinear evolution equations with a novel symbolic bilinear technique. This technique gave us an advantage in obtaining generalized soliton solutions depending on the arbitrary parameters and the constant presented in the phase variable for the investigated equations. We showed that the proposed technique establishes more generalized exact solutions than Hirota's N-solitons, which is a case with the parameter values. We investigated two well-known (1+1)-dimensional KdV and (2+1)-dimensional KP equations with the said technique and compared the obtained solutions to Hirota's soliton solutions. Generalized soliton solutions up to the third order are obtained, providing a better analysis and understanding of the solutions with arbitrary parameters. Dynamical analysis for the obtained generalized solitons has been shown through wave profiles with distinct values of the real parameters. The graphics for the first-order solution represented the single solitons. In contrast, the second and third-order solutions showed the solitons' interactions in X-type or Y-type interactions. These interactions change the positions depending on the choice of constant parameter present in phase shift. The physical significance of our research lies in the soliton solutions with more realistic and versatile arbitrary parameters. We have taken great care to ensure the validity of our results, validating them against the existing Hirota method. This validation process, along with the phase shift analysis, helps us better understand soliton interactions and provides a strong foundation for our findings. We have also discussed the phase shift and dispersion coefficient relations among arbitrary parameters, which verified the condition in Hirota's solitons solutions by choosing the values of arbitrary parameters as 1. Our analysis of the dynamic behavior of the obtained solutions with distinct parameter values, using the symbolic system *Mathematica*, further reinforces the reliability of our results.

As, the technique has been demonstrated on established equations like KdV and KP nonlinear models, there is still room for research to solve a more general class of nonlinear partial differential equations. Further research is needed to characterize how well this technique performs on computational problems that are more complex systems. These limitations are not roadblocks but point toward potential research directions. This technique offers the potential to provide generalized soliton solutions, it paves the way for significant advancements in research. The creation of generalized soliton solutions with variable parameters can enable a more adaptable and detailed explanation of physical systems, stimulating researchers and scientists intellectually. They can use this method to investigate and analyze a variety of evolution equations, leveraging the presence of arbitrary parameters to gain a deeper understanding in the fields of oceanography, plasma, fluid mechanics, water engineering, optical fibers, and other nonlinear systems.

CRediT authorship contribution statement

Brij Mohan: Writing – original draft, Validation, Software, Methodology, Investigation. **Sachin Kumar:** Writing – original draft, Supervision, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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ORIGINAL PAPER

Higher-order rogue waves and dispersive solitons of a novel P-type (3+1)-D evolution equation in soliton theory and nonlinear waves

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Abstract In soliton theory and nonlinear waves, this research proposes a new Painlevé integrable generalized (3+1)-D evolution equation. It demonstrates the Painlevé test that claims the integrability of the proposed equation and employs Cole–Hopf transformations to generate the trilinear equation in an auxiliary function that governs the higher-order rogue wave and dispersive-soliton solutions via the symbolic computation approach and dispersive-soliton assumption, respectively. Center-controlled parameters in rogue waves show the different dynamical structures with several other parameters. We obtain solutions for rogue waves up to third-order using direct symbolic analysis with appropriate center parameters and other parameters using a generalized procedure for rogue waves. We assume the dispersive-soliton solution, inspired by Hirota's direct techniques to create dispersive-soliton solutions up to the third order. By applying the symbolic software *Mathematica*, we demonstrate the dynamical structures for rogue waves with diverse center parameters and dispersive solitons using dispersion relation to showcase the interaction behavior of the solitons. Dispersive solitons and rogue waves are fascinating phenomena that appear in diverse areas of physics, such as optical fibers, nonlinear waves, dusty plasma physics, nonlinear dynamics, and other engineering and sciences.

Keywords Painlevé analysis · Cole–Hopf transformation · Generalized equation · Symbolic computational approach · Wave interactions

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1 Introduction

Dispersive solitons [1–4] are fascinating wave phenomena that appear in various areas of physics. Due to a careful balancing act between dispersion and nonlinearity, they stand out for their capacity to maintain their shape and stability across great distances. Dispersive solitons, sometimes called optical solitons or soliton pulses, are fundamental components of high-speed fiber optic communication systems. Solitons can traverse long distances without experiencing severe distortion by balancing the optical fiber's dispersion and the material's nonlinearity. This characteristic is essential for effective and trustworthy data transfer in optical communication networks. In hydrodynamics, dispersive solitons are single waves or solitons in water waves. The engineering of coastlines, oceanography, and wave energy conversion all use them in practical ways. Comprehending and governing these solitons can support addressing coastal erosion, improve wave forecasting, and optimize wave energy extraction. In addition, dispersive solitons, also known as plasma solitons or soliton waves, are observed in plasma physics. These solitons are produced due to the interaction between the plasma's dispersive characteristics and the nonlinearity brought on by particle interactions. They are helpful in several plasma physics research fields, such as investigating wave propagation in magnetized plasma, plasma heating, and fusion studies. Understanding and utilizing solitons in these fields has practical importance, ranging from high-speed data transmission in optical communication to coastal management and the advancement of quantum technologies.

The rogue waves [5–19], or localized substantial solitary waves in space-time, have a significant amplitude. These waves may cause people considerable harm and are unexpected. Numerous researchers are interested in exploring the way that rogue waves evolve. They stand out because of their abnormally steep height, often more than the neighboring waves. Rogue waves defy the predictions of traditional linear wave models, which is why nonlinear wave dynamics study them. Rogue wave study in nonlinear science seeks to understand their underlying physics and foretell their occurrence. One important use is enhancing maritime safety. Researchers can provide earlier detection and alarm systems to halt accidents carried out by rogue waves by developing algorithms or techniques for prediction models. This knowledge could be helpful in the marine industry, oil or gas outlets, and seaside infrastructure. Therefore, we may achieve improved functional security and cost-effective answers by understanding the dynamics of constructing secure systems and design techniques to decrease their consequences. Additionally, research into rogue wave dynamics advances the understanding of complex scenarios, interactions of waves, and how severe events arise in various physical phenomena.

This research proposes a new generalized (3+1)-D P-type equation in nonlinear waves as

$$u_{xxxy} + \alpha_1 u_{yt} + \alpha_2 (u u_x)_y + \alpha_3 u_{xx} + \alpha_4 u_{zz} = 0, \tag{1}$$

with α_i; $1 \leq i \leq 4$ as real coefficients. The nonlinear integrable equation carries localized solutions in specific directions, such as solitons, breathers, lumps, and other solutions. Analyzing the nonlinear partial differential equations (PDEs)' integrability can lead to precise and analytical solutions. A nonlinear PDE can have its entire integrability verified by the Painlevé test [20–24]. It becomes pretty laborious to determine whether a PDE meets the Painlevé test. However, this analysis is possible because of symbolic tools, such as *Mathematica* and *Matlab* software. We search for specific solutions to correctly comprehend the peculiarities of different facts in diverse natural science domains. As was already said, nonlinear PDE has attracted the interest of various academics because it closely reflects real-world situations and offers a vast range of solutions. Symbolic system software can help to encounter these solutions quickly. In 2006, Baldwin and Hereman [20] developed a computation in Mathematica to perform the WTC-Kruskal approach-based Painlevé test for PDEs. It has been an attractive study area to highlight fundamental concepts in nonlinear sciences, plasmas, water engineering, and shallow water waves through dynamically analyzing rogue wave behavior produced by nonlinear PDEs.

A vast area of mathematics and physics known as nonlinear PDEs deals with nonlinear functions that represent the models for complex physical systems in various research fields. Mathematicians have used nonlinear PDEs to showcase several physical phenomena, from nonlinear dynamics to gravity, and solve problems like Poincare's and Calabi's conjectures. Since only some general methods can be used to analyze nonlinear PDEs, they are challenging to study. Each equation must typically be studied as a separate problem. However, there are some situations when general methods can be applied. These techniques discretize the domain of the issue into a grid of points and then utilize mathematical approaches to estimate the solution. For obtaining the analytical and exact solutions, several techniques are being utilized, including the streamlined Hirota's approach [25,26], the Bäcklund transformation [27,28], Bilinear Neural Network Method [1,29–31], the Hirota's bilinearization method [32–35], Lie symmetry analysis [36–39], and the Darboux transformation [40–44].

The manuscript is structured as: The next section examines the proposed generalized equation for being integrable by utilizing the test of Painlevé. Section 3 illustrates the Cole–Hopf transformation and symbolic computation to get the rogue waves up to the third order. In this computation, we utilize the dependent variable transformation to change the equation in a trilinear equation by finding the dispersion through the phase. It showcases the dynamics graphically for the obtained solutions. In Sect. 4, we get the dispersive-soliton solutions up to the third order and portray the dynamical structures with chosen distinct parameters. Section 5 concerns the research findings of generated solutions, and the end section discusses the research conclusion.

2 P-integrability analysis

Painlevé test is a robust tool for investigating the integrability of nonlinear PDEs. It bears the name of French mathematician Paul Painlevé, who contributed substantially to studying nonlinear equations. Determining whether a given nonlinear PDE admits solutions devoid of movable singularities is the main objective of this analysis. A singularity that can be eliminated or altered by an appropriate coordinate transformation is movable. A PDE is considered P-integrable if it passes the Painlevé test, which denotes that it has a complex structure and may be solved using specialized functions. Assuming a particular form for the PDE solutions as part of Painlevé analysis entails expanding the Laurent series around a movable singularity. A set of consistency conditions is generated by inserting these presumptive forms into the PDE and equating the coefficients of like powers. If the test is unsuccessful, it suggests that the PDE might have singularities that are not generic or lack Painlevé integrability. It has shown to be an effective method for analyzing nonlinear PDEs since it sheds light on the presence of unique solutions and reveals integrable structures that were previously concealed. Researchers can create analytical strategies and methods to solve these equations and better comprehend the underlying physical phenomena they represent by identifying integrable PDEs.

Weiss with Tabor and Carnevale (WTC) [22] gave the Painlevé test to examine the nonlinear PDEs for integrability by confirming integrable conditions. This analysis has three stages: first, it examines the leading-order behavior; second, it locates the resonances; and third, it confirms the conditions for resonances. The test considers P-integrable if all the movable singularities are the simple poles of the solutions. A Laurent's series with analytical function g, about the singular manifold $g=0$, expands the field u as

$$u=\sum_{r=0}^{\infty} u_r g^{r+\Lambda}, \tag{2}$$

with Λ and u_r; $r=0,1,2,\ldots$; as the integer and arbitrary functions, respectively. On putting Eq. (2) into (1), we get Λ by leading order analysis as

$$\Lambda=-2,$$

and retrieve

$$u_0=-\frac{12g_{\mathrm{x}}^2}{\alpha_2}.$$

We get the characteristic equation for the resonances as

$$(r+1)(r-4)(r-5)(r-6)\alpha_2 g_y g_x^3=0, \tag{3}$$

which gives the resonances

$$r=-1,\quad 4,\quad 5,\quad 6.$$

Due to the arbitrary choice of $g=0$, we get resonance $r=-1$. Functions u_r exist explicitly for $r=0,1,2,3$ and arbitrary choices for $r=4,5,6$, are given as

$$u_1=\frac{12g_{\mathrm{xx}}}{\alpha_2},$$

$$u_2=-\frac{\alpha_1 g_{\mathrm{t}} g_{\mathrm{x}} g_{\mathrm{y}}+\alpha_3 g_{\mathrm{x}}^3+4g_{\mathrm{x}} g_{\mathrm{xxx}} g_{\mathrm{y}}+\alpha_4 g_{\mathrm{x}} g_{\mathrm{z}}^2-3g_{\mathrm{xx}}^2 g_{\mathrm{y}}}{\alpha_2 g_{\mathrm{x}}^2 g_{\mathrm{y}}},$$

$$u_3=\frac{-\alpha_1 g_{\mathrm{t}} g_{\mathrm{x}} g_{\mathrm{xx}} g_{\mathrm{y}}^3+\alpha_1 g_{\mathrm{x}}^2 g_{\mathrm{xt}} g_{\mathrm{y}}^3-4g_{\mathrm{x}} g_{\mathrm{xx}} g_{\mathrm{xxx}} g_{\mathrm{y}}^3+\cdots+\alpha_4 g_{\mathrm{x}}^3 g_{\mathrm{yy}} g_{\mathrm{z}}^2+3g_{\mathrm{xx}}^3 g_{\mathrm{y}}^3}{\alpha_2 g_{\mathrm{x}}^4 g_{\mathrm{y}}^3},$$

$$u_4=u_4,\qquad u_5=u_5,\qquad u_6=u_6,$$

where expression of u_3 is lengthy, so middle terms are skipped. The resonances $r=4,5,6$ identically fulfill the conditions for compatibility. Therefore, the proposed Eq. (1) is integrable entirely or satisfies the P-integrability.

3 Cole–Hopf transformation and rogue waves

By applying the transformations $U = x + t$ and $V = y + z$ for $u = u(U, V)$ in Eq. (1), we get

$$\alpha_1 u_{UV} + \alpha_2(u_V u_U + u u_{UV}) + \alpha_3 u_{UU} + \alpha_4 u_{VV} + u_{UUUV} = 0. \tag{4}$$

We consider the phase Φ_i in Eq. (4) as

$$\Phi_i = p_i U - w_i V, \tag{5}$$

with w_i as the dispersion-coefficient and p_i as real-parameter. By taking $u(U, V) = e^{\Phi_i}$ into the terms with linearity of Eq. (4), we get

$$w_i = \frac{p_i^3 + \alpha_1 p_i \pm p_i\sqrt{\alpha_1^2 - 4\alpha_3\alpha_4 + 2\alpha_1 p_i^2 + p_i^4}}{2\alpha_4}. \tag{6}$$

Now, taking the transformation as

$$u(U, V) = R(\ln f)_{UU}, \tag{7}$$

where R and f are the constant and auxiliary function, respectively. We put this transformation with Eq. (6) and $f(U, V) = 1 + e^{\Phi_1}$ into Eq. (4), and solve for R which gives

$$R = \frac{12}{\alpha_2}.$$

Thus, the transformation (7) becomes

$$u(U, V) = \frac{12}{\alpha_2}(lnf)_{UU}. \tag{8}$$

On substituting Eq. (8) into Eq. (4) gives a trilinear equation in $f(U, V)$ as

$$\begin{aligned}&\alpha_4 f^2 f_{UVV} + \alpha_1 f^2 f_{UUV} + \alpha_3 f^2 f_{UUU}\\ &+ f^2 f_{UUUUV} + 2\alpha_1 f_V f_U^2 + 2\alpha_4 f_V^2 f_U\\ &- 2\alpha_4 f f_V f_{UV} - \alpha_1 f f_V f_{UU}\\ &- \alpha_4 f f_{VV} f_U - 2\alpha_1 f f_U f_{UV} - 3\alpha_3 f f_U f_{UU}\\ &+ 2\alpha_3 f_U^3 + 8 f_V f_U f_{UUU} - 6 f_V f_{UU}^2\\ &- f f_V f_{UUUU}\\ &- 4 f f_U f_{UUUV} - 4 f f_{UV} f_{UUU}\\ &+ 6 f f_{UU} f_{UUV} = 0.\end{aligned} \tag{9}$$

We consider the function f for the rogue waves solutions [45–48] as

$$\begin{aligned}f(U, V) &= \widehat{F_N}(U, V, \beta, \gamma)\\ &= \sum_{s=0}^{\frac{N(N+1)}{2}} \sum_{i=0}^{s} c_{N(N+1)-2s,2i}(V-\gamma)^{2i}\\ &\quad (U-\beta)^{N(N+1)-2s},\end{aligned} \tag{10}$$

with $c_{p,q}$; $p, q \in \{0, 2, \cdots, s(s+1)\}$ as the constants; β, γ as parameters controlling the center.

3.1 First-order rogue waves

To obtain first-order rogue wave, take the function $f(U, V)$ from Eq. (10) with $N = 1$ as

$$f(U, V) = c_{0,0} + c_{0,2}V^2 + c_{2,0}U^2. \tag{11}$$

On putting Eq. (11) into Eq. (9), we get a system of equations by equating zero the coefficients of different powers of $U^p V^q$; $p, q \in Z$, as

$$\begin{aligned}&\alpha_4 c_{0,2}^2 c_{2,0} - \alpha_3 c_{0,2} c_{2,0}^2 = 0,\\ &\alpha_1 c_{0,0} c_{0,2} c_{2,0} + 12 c_{0,2} c_{2,0}^2 = 0,\\ &3\alpha_3 c_{0,0} c_{2,0}^2 + \alpha_4 c_{0,0} c_{0,2} c_{2,0} = 0,\end{aligned} \tag{12}$$

which gives the constant values as

$$c_{0,0} = -\frac{12 c_{2,0}}{\alpha_1}, \quad c_{0,2} = \frac{\alpha_3 c_{2,0}}{\alpha_4}, \quad c_{2,0} = c_{2,0}. \tag{13}$$

Therefore, Eq. (11) with values (13) becomes

$$\begin{aligned}f(U, V) &= \widehat{F}_1(U, V, \beta, \gamma)\\ &= c_{2,0}\left(\frac{\alpha_3(\gamma - V)^2}{\alpha_4} - \frac{12}{\alpha_1} + (\beta - U)^2\right),\end{aligned} \tag{14}$$

which is a solution of Eq. (9). On putting Eq. (14) into (8), we obtain first-order rogue wave solution as

$$u(U, V) = -\frac{24\alpha_1\alpha_4\left(\alpha_1\left(\alpha_4(\beta-U)^2 - \alpha_3(\gamma-V)^2\right) + 12\alpha_4\right)}{\alpha_2\left(\alpha_1\left(\alpha_4(\beta-U)^2 + \alpha_3(\gamma-V)^2\right) - 12\alpha_4\right)^2}. \tag{15}$$

3.2 Second-order rogue waves

We assume the function $f(U, V)$ for $N = 2$ in Eq. (10) as

$$\begin{aligned}f(U, V) &= c_{0,0} + c_{0,2}V^2 + c_{0,4}V^4 + c_{0,6}V^6\\ &+ c_{2,0}U^2 + c_{2,2}V^2U^2 + c_{2,4}V^4U^2\\ &+ c_{4,0}U^4 + c_{4,2}V^2U^4 + c_{6,0}U^6.\end{aligned} \tag{16}$$

By putting Eq. (16) in trilinear Eq. (9), and taking zero the coefficients of distinct powers of $U^p V^q$; $p, q \in Z$, gets a system. This system gives the constants on solving as

$$c_{0,0} = \frac{76032\alpha_4 c_{4,2}}{61\alpha_1^3\alpha_3}, \quad c_{0,2} = -\frac{2688 c_{4,2}}{61\alpha_1^2},$$

$$c_{0,4} = -\frac{20\alpha_3 c_{4,2}}{\alpha_1\alpha_4},$$
$$c_{0,6} = \frac{\alpha_3^2 c_{4,2}}{3\alpha_4^2}, \quad c_{2,0} = \frac{23808\alpha_4 c_{4,2}}{61\alpha_1^2\alpha_3}, \quad c_{2,2} = \frac{24c_{4,2}}{\alpha_1},$$
$$c_{2,4} = \frac{\alpha_3 c_{4,2}}{\alpha_4}, \quad c_{4,0} = \frac{28\alpha_4 c_{4,2}}{\alpha_1\alpha_3},$$
$$c_{6,0} = \frac{\alpha_4 c_{4,2}}{3\alpha_3}, \tag{17}$$

with $c_{4,2}$ as an arbitrary constant. Therefore, Eq. (11) becomes

$$\begin{aligned} f(U,V) &= \widehat{F}_2(U,V,\beta,\gamma) \\ &= \frac{c_{4,2}}{183}\Bigg(\frac{183\alpha_3(\beta-U)^2(\gamma-V)^4}{\alpha_4} \\ &\quad +\frac{4392(\beta-U)^2(\gamma-V)^2}{\alpha_1}+\frac{61\alpha_4(\beta-U)^6}{\alpha_3}+ \\ &\quad \frac{5124\alpha_4(\beta-U)^4}{\alpha_1\alpha_3}+\frac{71424\alpha_4(\beta-U)^2}{\alpha_1^2\alpha_3} \\ &\quad +\frac{61\alpha_3^2(\gamma-V)^6}{\alpha_4^2}-\frac{3660\alpha_3(\gamma-V)^4}{\alpha_1\alpha_4} \\ &\quad -\frac{8064(\gamma-V)^2}{\alpha_1^2}+\frac{228096\alpha_4}{\alpha_1^3\alpha_3} \\ &\quad +183(\beta-U)^4(\gamma-V)^2\Bigg), \end{aligned} \tag{18}$$

is a solution of Eq. (9). On putting Eq. (18) into (8), gives a second-order solution of rogue wave as

$$u(U,V) = \frac{12}{\alpha_2}(\ln \widehat{F}_2(U,V,\beta,\gamma))_{UU}. \tag{19}$$

3.3 Third-order rogue waves

Taking $f(U,V)$ for $N=3$ in Eq. (10) to obtain third-order rogue wave solution as

$$\begin{aligned} f(U,V) &= c_{0,0}+c_{0,2}V^2+c_{0,4}V^4+c_{0,6}V^6 \\ &+c_{0,8}V^8+c_{0,10}V^{10}+c_{0,12}V^{12} \\ &+c_{2,0}U^2+c_{2,2}V^2U^2+c_{2,4}V^4U^2 \\ &+c_{2,6}V^6U^2+c_{2,8}V^8U^2+c_{2,10}V^{10}U^2 \\ &+c_{4,0}U^4+c_{4,2}V^2U^4+c_{4,4}V^4U^4+c_{4,6}V^6U^4 \\ &+c_{4,8}V^8U^4+c_{6,0}U^6 \\ &+c_{6,2}V^2U^6+c_{6,4}V^4U^6+c_{6,6}V^6U^6 \\ &+c_{8,0}U^8+c_{8,2}V^2U^8+c_{8,4}V^4U^8+c_{10,0}U^{10} \\ &+c_{10,2}V^2U^{10}+c_{12,0}U^{12}. \end{aligned} \tag{20}$$

Putting (20) in trilinear Eq. (9), and taking zero the coefficients of distinct powers of U^pV^q; $p,q \in Z$, gives a system of equations. We get the constant values on solving it as

$$c_{0,0} = \frac{6140529544625285373\alpha_4 c_{10,2}}{138083523968\alpha_1^6\alpha_3},$$
$$c_{0,2} = -\frac{171277579856366937c_{10,2}}{2655452384\alpha_1^5},$$
$$c_{0,4} = -\frac{2623638233493\alpha_3 c_{10,2}}{578656\alpha_1^4\alpha_4},$$
$$c_{0,6} = -\frac{1387653\alpha_3^2 c_{10,2}}{676\alpha_1^3\alpha_4^2}, \quad c_{0,8} = \frac{42129\alpha_3^3 c_{10,2}}{104\alpha_1^2\alpha_4^3},$$
$$c_{0,10} = -\frac{31\alpha_3^4 c_{10,2}}{2\alpha_1\alpha_4^4},$$
$$c_{0,12} = \frac{\alpha_3^5 c_{10,2}}{6\alpha_4^5}, \quad c_{2,0} = \frac{650903203872153\alpha_4 c_{10,2}}{2655452384\alpha_1^5\alpha_3},$$
$$c_{2,2} = \frac{1828730331c_{10,2}}{1712\alpha_1^4}, \quad c_{2,4} = -\frac{1672605\alpha_3 c_{10,2}}{676\alpha_1^3\alpha_4},$$
$$c_{2,6} = \frac{32829\alpha_3^2 c_{10,2}}{26\alpha_1^2\alpha_4^2}, \quad c_{2,8} = -\frac{93\alpha_3^3 c_{10,2}}{2\alpha_1\alpha_4^3},$$
$$c_{2,10} = \frac{\alpha_3^4 c_{10,2}}{\alpha_4^4},$$
$$c_{4,0} = -\frac{19319920533\alpha_4 c_{10,2}}{578656\alpha_1^4\alpha_3}, \quad c_{4,2} = \frac{1672605c_{10,2}}{676\alpha_1^3},$$
$$c_{4,4} = \frac{89187\alpha_3 c_{10,2}}{52\alpha_1^2\alpha_4},$$
$$c_{4,6} = -\frac{31\alpha_3^2 c_{10,2}}{\alpha_1\alpha_4^2}, \quad c_{4,8} = \frac{5\alpha_3^3 c_{10,2}}{2\alpha_4^3},$$
$$c_{6,0} = \frac{1387653\alpha_4 c_{10,2}}{676\alpha_1^3\alpha_3},$$
$$c_{6,2} = \frac{32829c_{10,2}}{26\alpha_1^2}, \quad c_{6,4} = \frac{31\alpha_3 c_{10,2}}{\alpha_1\alpha_4},$$
$$c_{6,6} = \frac{10\alpha_3^2 c_{10,2}}{3\alpha_4^2},$$
$$c_{8,0} = \frac{42129\alpha_4 c_{10,2}}{104\alpha_1^2\alpha_3}, \quad c_{8,2} = \frac{93c_{10,2}}{2\alpha_1},$$
$$c_{8,4} = \frac{5\alpha_3 c_{10,2}}{2\alpha_4}, \quad c_{10,0} = \frac{31\alpha_4 c_{10,2}}{2\alpha_1\alpha_3},$$
$$c_{12,0} = \frac{\alpha_4 c_{10,2}}{6\alpha_3}, \tag{21}$$

with $c_{10,2}$ as an arbitrary constant. Therefore, Eq. (11) becomes

$$f(U,V) = \widehat{F}_3(U,V,\beta,\gamma) = \frac{c_{10,2}}{414250571904\alpha_1^6\alpha_3\alpha_4^5}$$
$$+(18421588633875856119\alpha_4^6$$

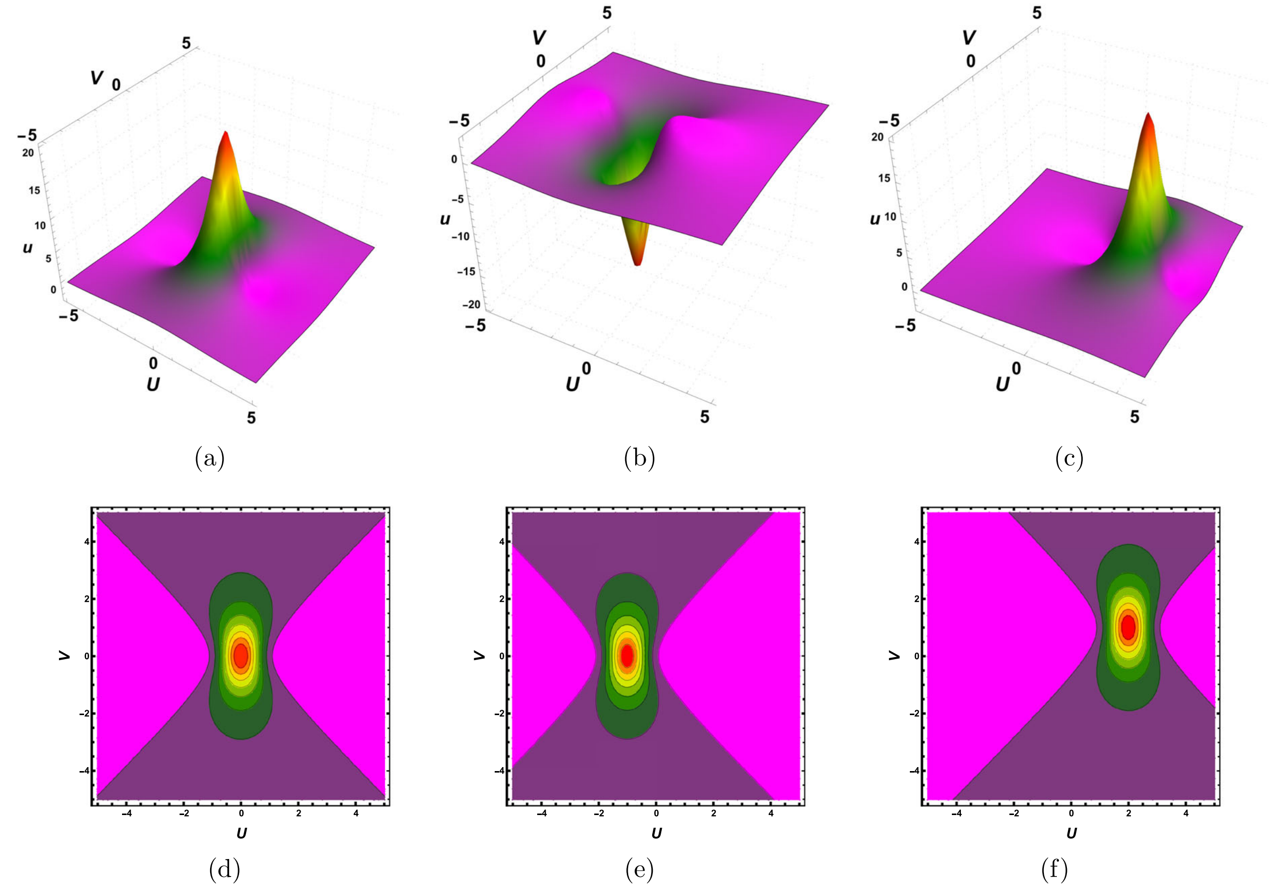

Fig. 1 First-order rogue waves of (15) for (14) with values: $\alpha_1 = -10$, $\alpha_2 = \alpha_3 = \alpha_4 = 1$, and center parameters (β, γ) as: **a** (0,0); **b** (−1,0); and **c** (2,1). **d–f** are 2-D contours for **a–c** in UV-plane

$$
\begin{aligned}
&+690417761984\alpha_1^6\alpha_4^6P^{12}\\
&+6420883864512\alpha_1^5\alpha_4^6P^{10}\\
&+414250571904\alpha_1^6\alpha_3\alpha_4^5P^{10}Q^2\\
&+167807330228304\alpha_1^4\alpha_4^6P^8\\
&+1035626429760\alpha_1^6\alpha_3^2\alpha_4^4P^8Q^4\\
&+19262651593536\alpha_1^5\alpha_3\alpha_4^5P^8Q^2\\
&+850349184695712\alpha_1^3\alpha_4^6P^6\\
&+1380835239680\alpha_1^6\alpha_3^3\alpha_4^3P^6Q^6\\
&+12841767729024\alpha_1^5\alpha_3^2\alpha_4^4P^6Q^4\\
&+523055077886016\alpha_1^4\alpha_3\alpha_4^5P^6Q^2\\
&-1383082190846172\alpha_1^2\alpha_4^6P^4\\
&+1035626429760\alpha_1^6\alpha_3^4\alpha_4^2P^4Q^8\\
&-12841767729024\alpha_1^5\alpha_3^3\alpha_4^3P^4Q^6\\
&+710495495315424\alpha_1^4\alpha_3^2\alpha_4^4P^4Q^4\\
&+1024966831093920\alpha_1^3\alpha_3\alpha_4^5P^4Q^2\\
&+101540899804055868\alpha_1\alpha_4^6P^2\\
&+414250571904\alpha_1^6\alpha_3^5\alpha_4P^2Q^{10}\\
&-19262651593536\alpha_1^5\alpha_3^4\alpha_4^2P^2Q^8\\
&+523055077886016\alpha_1^4\alpha_3^3\alpha_4^3P^2Q^6\\
&-1024966831093920\alpha_1^3\alpha_3^2\alpha_4^4P^2Q^4\\
&+442495669085830152\alpha_1^2\alpha_3\alpha_4^5P^2Q^2\\
&+69041761984\alpha_1^6\alpha_3^6Q^{12}\\
&-6420883864512\alpha_1^5\alpha_3^5\alpha_4Q^{10}\\
&+167807330228304\alpha_1^4\alpha_3^4\alpha_4^2Q^8\\
&-850349184695712\alpha_1^3\alpha_3^3\alpha_4^3Q^6\\
&-1878220633145902812\alpha_1^2\alpha_3^2\alpha_4^4Q^4\\
&-26719302457593242172\alpha_1\alpha_3\alpha_4^5Q^2),
\end{aligned}
\tag{22}
$$

where $P = (\beta - U)$ and $Q = (\gamma - V)$, that is a solution of Eq. (9) with center parameters (β, γ). Putting Eq.

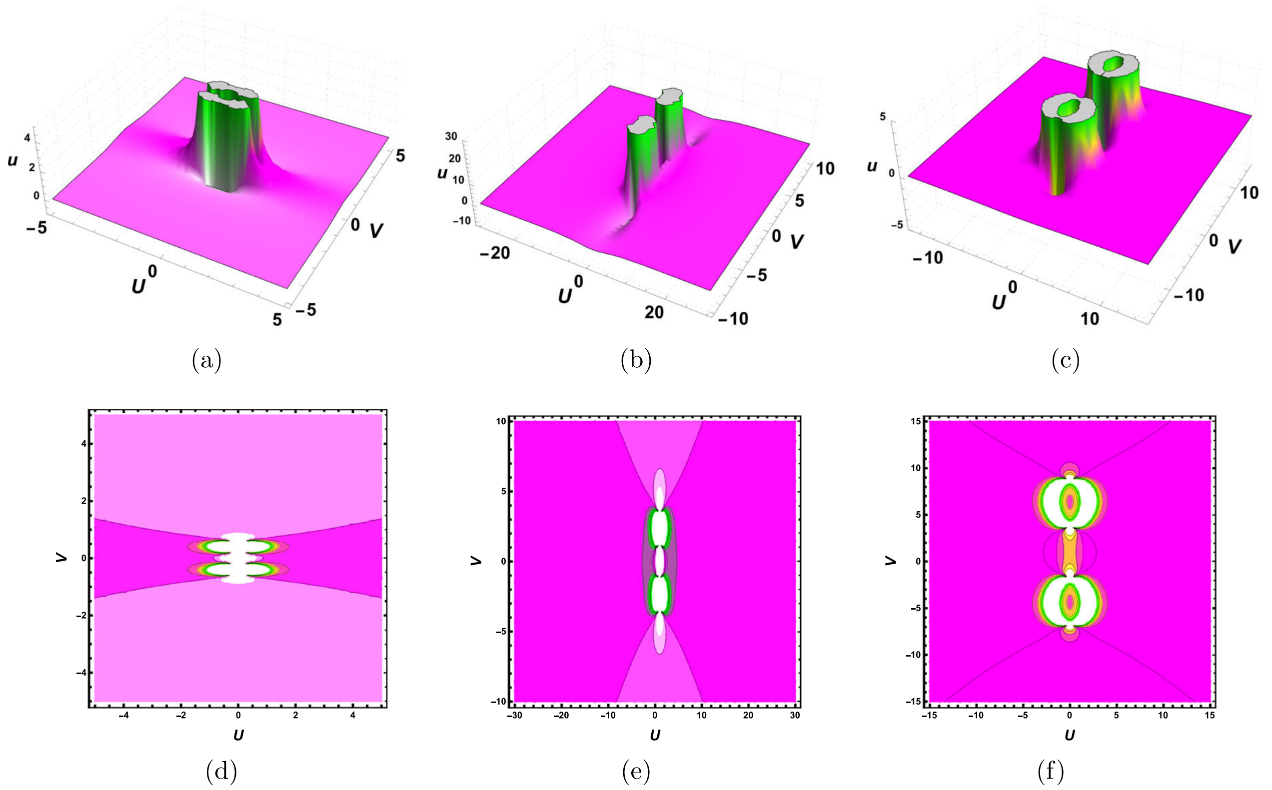

Fig. 2 Second-order rogue waves of (19) for (18) with values: **a** $\alpha_1 = 10, \alpha_2 = -10, \alpha_3 = 35, \alpha_4 = 2, \beta = \gamma = 0$; **b** $\alpha_1 = 5, \alpha_2 = -0.5, \alpha_3 = \alpha_4 = 1, \beta = 1, \gamma = 6$; and **c** $\alpha_1 = 1, \alpha_2 = -3, \alpha_3 = \alpha_4 = 0.01, \beta = 0, \gamma = 1$; and **d-f** are contours w.r.t. **a**–**c** in UV-plane

(22) into (8) gives a third-order solution of rogue wave as

$$u(U, V) = \frac{12}{\alpha_2}(\ln \widehat{F}_3(U, V, \beta, \gamma))_{UU}. \tag{23}$$

4 Logarithmic transformation and dispersive solitons

We take Φ_i as the phase in Eq. (1) as

$$\Phi_i = p_i x + q_i y + r_i z - w_i t, \tag{24}$$

where w_i and p_i, q_i, r_i with $i = 1, 2, 3, \ldots$, are the coefficients of dispersion and real parameters, respectively. Having $u = e^{\Phi_i}$ into the linear terms of (1) gives dispersion as

$$w_i = \frac{\alpha_3 p_i^2 + p_i^3 q_i + \alpha_4 r_i^2}{\alpha_1 q_i}. \tag{25}$$

Now, we assume the logarithmic transformation

$$u(x, y, z, t) = R(\ln f)_{xx}, \tag{26}$$

and put it with Eq. (25) and $f(U, V) = 1 + e^{\Phi_1}$ into Eq. (1). Thus, solving for R gives

$$R = \frac{12}{\alpha_2}.$$

The transformation (26) becomes

$$u(x, y, z, t) = \frac{12}{\alpha_2}(\ln f)_{xx}. \tag{27}$$

Putting the transformation (27) into Eq. (1) gives a trilinear equation in $f(x, y, z, t)$ as

$$\begin{aligned}
&\alpha_3 f^2 f_{xxx} + f^2 f_{xxxxy} + \alpha_1 f^2 f_{xyt} + \alpha_4 f^2 f_{xzz} \\
&\quad + 2\alpha_1 f_t f_x f_y - \alpha_1 f f_t f_{xy} + 2\alpha_3 f_x^3 - 3\alpha_3 f f_x f_{xx} \\
&\quad + 8 f_x f_{xxx} f_y - 4 f f_x f_{xxxy} - \alpha_1 f f_x f_{yt} + 2\alpha_4 f_x f_z^2 \\
&\quad - \alpha_4 f f_x f_{zz} - \alpha_1 f f_{xt} f_y + 6 f f_{xx} f_{xxy} \\
&\quad - 6 f_{xx}^2 f_y - 4 f f_{xxx} f_{xy} - f f_{xxxx} f_y \\
&\quad - 2\alpha_4 f f_{xz} f_z = 0.
\end{aligned} \tag{28}$$

To obtain dispersive-soliton solution, assume a closed-form expression for the function f as N dispersive-

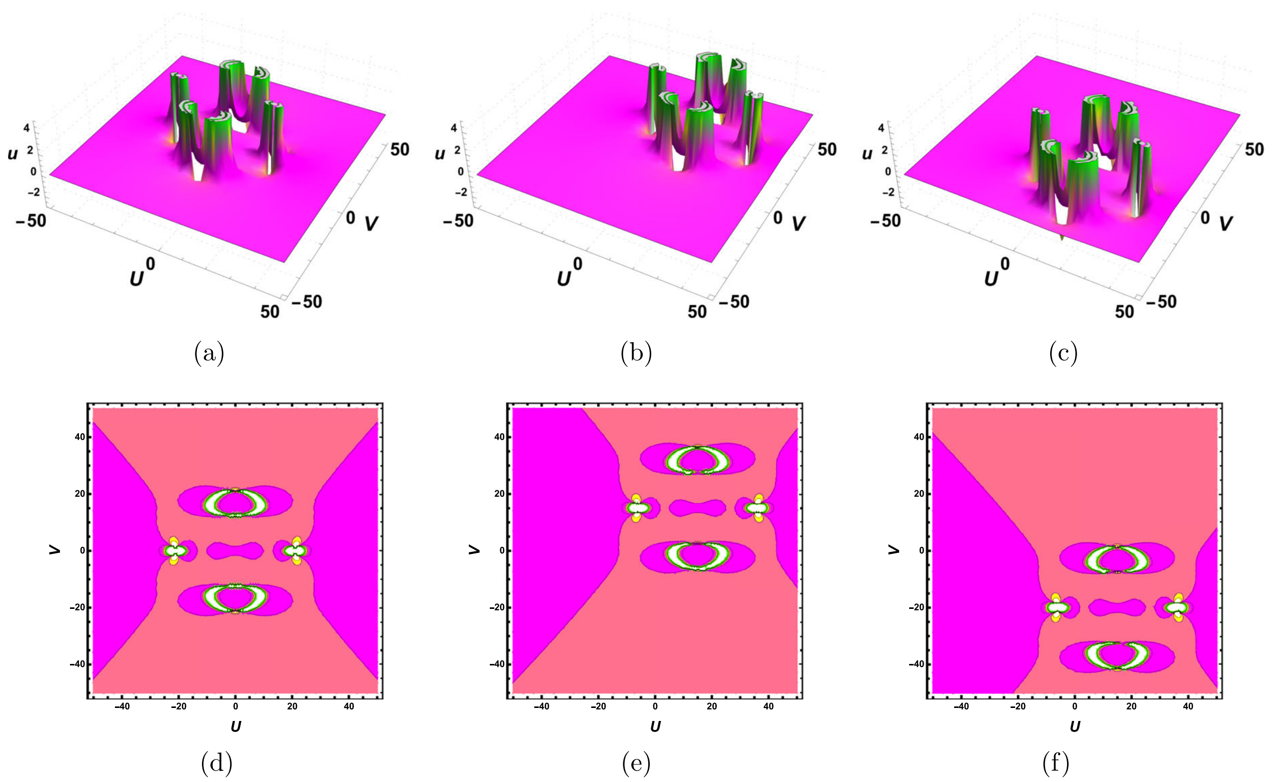

Fig. 3 Third-order rogue waves of (23) for (22) with values: $\alpha_1 = -0.1, \alpha_2 = -1, \alpha_3 = 0.5, \alpha_4 = 0.5$, and parameters (β, γ) as: **a** (0,0); **b** (15,15); and **c** (15,−20). **d–f** are contours w.r.t. **a–c** in UV-plane

soliton inspired by Hirota [32], in transformation (27) as

$$f = \sum_{\lambda=0,1} e^{\left(\sum_{1\le a\le N} \lambda_a \Phi_a + \sum_{1\le a<b\le N} \lambda_a \lambda_b\right)}, \tag{29}$$

where $\sum_{\lambda=0,1}$ is the addition of all possible combinations for $\lambda_n = 0, 1$ for $1 \le n \le N$.
We get two choices $\lambda_1 = 0, 1$ for $N = 1$, so $f = 1 + e^{\Phi_1}$,
and have $\lambda_1 = 0, 1$ and $\lambda_2 = 0, 1$ for $N = 2$, So there will be four combinations of λ_1 and λ_2, thus the function f will be as

$$f = 1 + e^{\Phi_1} + e^{\Phi_2} + e^{\Phi_1+\Phi_2},$$

similarly, $\lambda_1, \lambda_2, \lambda_3 = 0, 1$ for $N = 3$, so the total combinations for λ_1, λ_2, and λ_3 will be eight. Thus, the expression for f will be as

$$f = 1 + e^{\Phi_1} + e^{\Phi_2} + e^{\Phi_3} + e^{\Phi_1+\Phi_2} + e^{\Phi_1+\Phi_3} + e^{\Phi_2+\Phi_3} + e^{\Phi_1+\Phi_2+\Phi_3}.$$

4.1 Single-dispersive-soliton solution

Assuming the auxiliary function f in Eq. (28) as

$$f(x, y, z, t) = 1 + e^{\Phi_1} = 1 + e^{p_1 x + q_1 y + r_1 z - w_1 t}. \tag{30}$$

Equation (30) gives

$$f_x = p_1 e^{p_1 x + q_1 y + r_1 z - w_1 t}, \tag{31}$$

$$f_{xx} = p_1^2 e^{p_1 x + q_1 y + r_1 z - w_1 t}. \tag{32}$$

On substituting (30), (31) and (32) into Eq. (27), a single-dispersive-soliton solution is obtained as

$$u(x, y, z, t) = \frac{12 p_1^2 \exp\left(\frac{t(\alpha_3 p_1^2 + p_1^3 q_1 + \alpha_4 r_1^2)}{\alpha_1 q_1} + p_1 x + q_1 y + r_1 z\right)}{\alpha_2 \left(\exp\left(\frac{t(\alpha_3 p_1^2 + p_1^3 q_1 + \alpha_4 r_1^2)}{\alpha_1 q_1}\right) + \exp(p_1 x + q_1 y + r_1 z)\right)^2}. \tag{33}$$

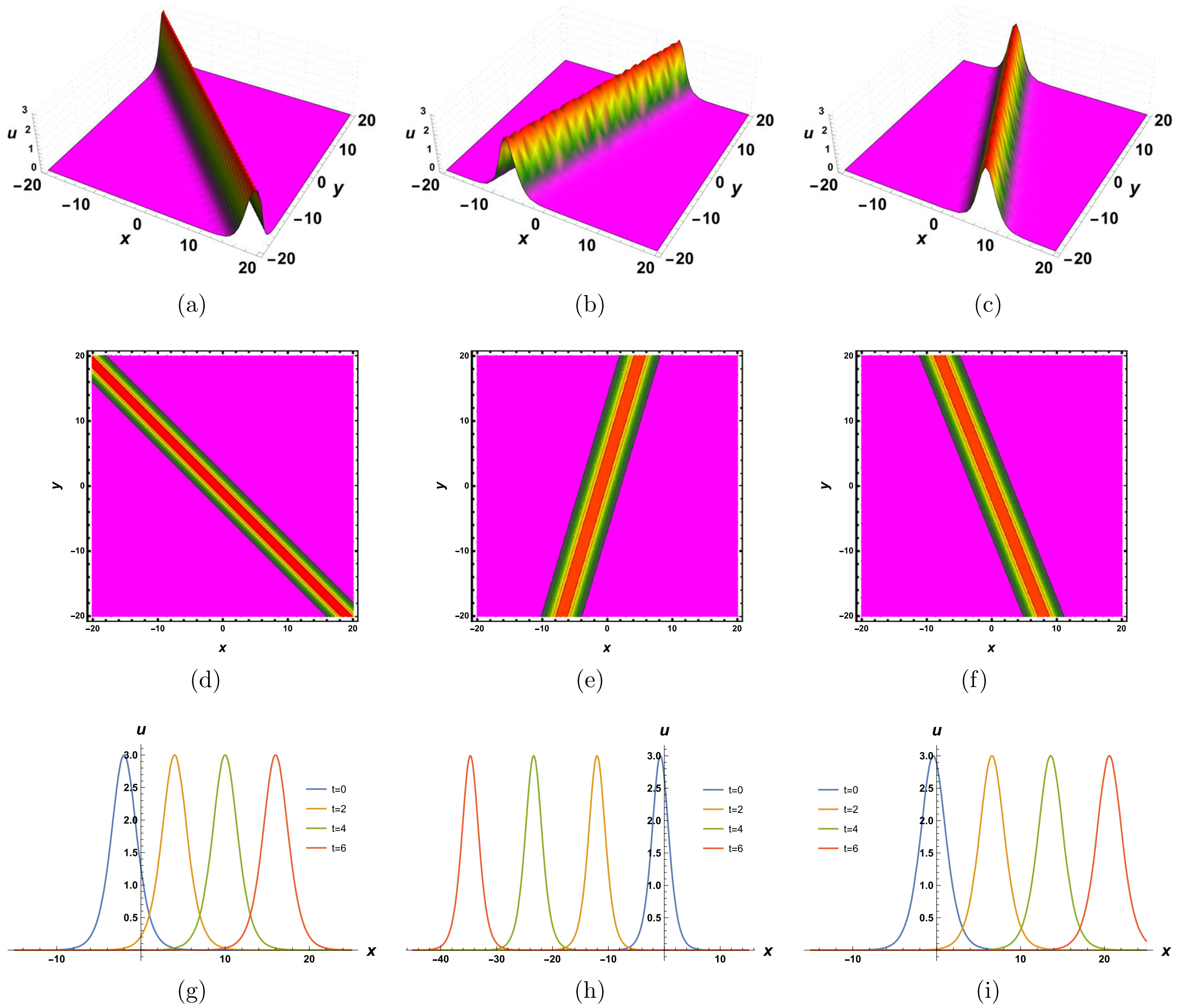

Fig. 4 Single dispersive soliton for (33) with (30) having values: **a** $p_1 = q_1 = r_1 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, t = 0, z = 1$; **b** $p_1 = 1, q_1 = -0.3, r_1 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, t = 0, z = 1$ and **c** $p_1 = 1, q_1 = -0.4, r_1 = 0, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, t = 0, z = 1$. **d-f** and **g-i** are the 2-D contours with $y = 1$, for **a-c**, respectively

4.2 Two-dispersive-soliton solution

To get a two-dispersive-soliton solution, assume f as

$$f(x, y, z, t) = 1 + e^{\Phi_1} + e^{\Phi_2} + e^{\Phi_1+\Phi_2}, \tag{34}$$

Which gives

$$f_x = p_1 e^{\Phi_1} + p_2 e^{\Phi_2} + (p_1 + p_2) e^{\Phi_1+\Phi_2}, \tag{35}$$

$$f_{xx} = p_1^2 e^{\Phi_1} + p_2^2 e^{\Phi_2} + (p_1 + p_2)^2 e^{\Phi_1+\Phi_2}. \tag{36}$$

Substituting Eqs. (34), (35) and (36) into (27) gives a two-dispersive-soliton solution of equation (1) as

$$u(x, y, z, t) = \frac{12\left(p_1^2 \exp\left(\frac{t(Q_1)}{\alpha_1 q_1} + P_1\right)\left(\exp\left(\frac{t(Q_2)}{\alpha_1 q_2}\right) + e^{P_2}\right)^2 + p_2^2 \exp\left(\frac{t(Q_2)}{\alpha_1 q_2} + P_2\right)\left(\exp\left(\frac{t(Q_1)}{\alpha_1 q_1}\right) + e^{P_1}\right)^2\right)}{\alpha_2\left(\exp\left(\frac{t(Q_1)}{\alpha_1 q_1}\right) + e^{P_1}\right)^2\left(\exp\left(\frac{t(Q_2)}{\alpha_1 q_2}\right) + e^{P_2}\right)^2}, \tag{37}$$

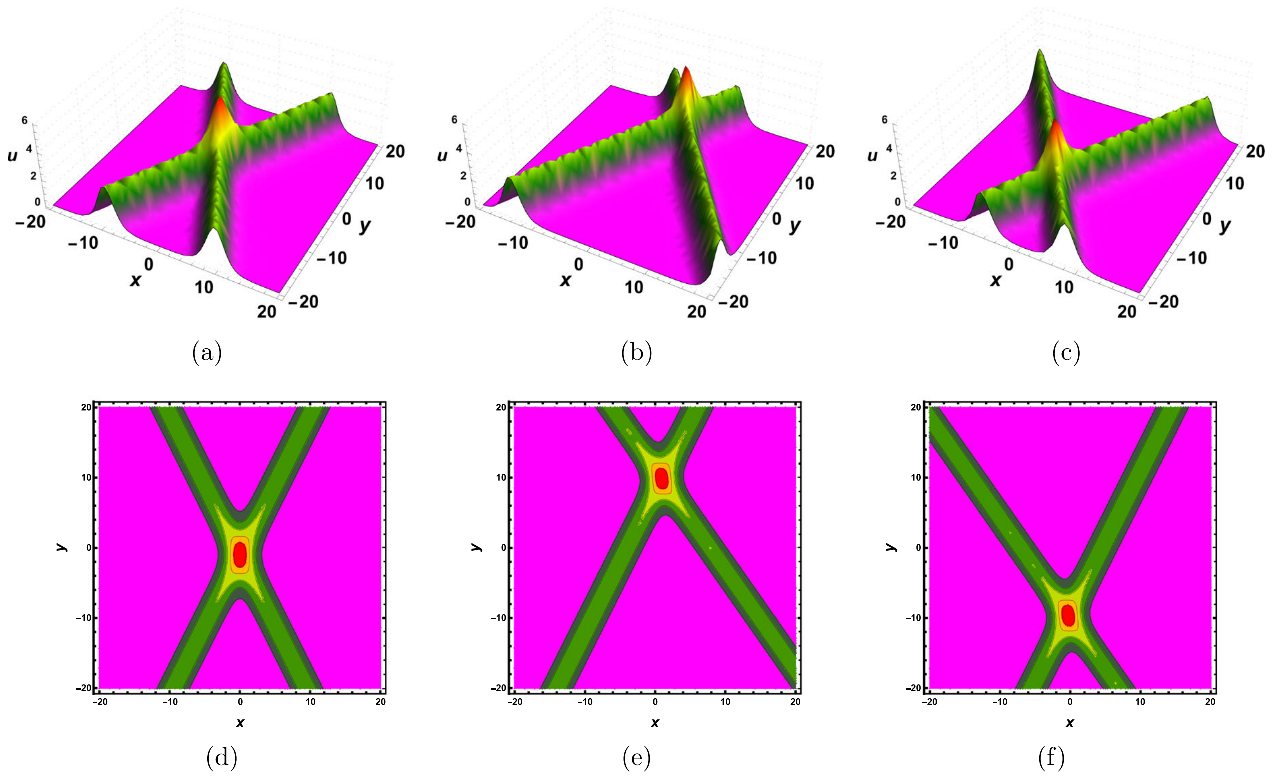

Fig. 5 Interaction of two dispersive solitons for (37) with (34) having values: **a** $p_1 = 1, q_1 = r_1 == 0.5, p_2 = -1, q_2 = r_2 = 0.5, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, t = 0, z = 1$; **b** $p_1 = 1, q_1 = 0.7, r_1 = 0.5, p_2 = -1, q_2 = r_2 = 0.5, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, t = 3, z = 1$; and **c** $p_1 = 1, q_1 = 0.7, r_1 == 0.5, p_2 = -1, q_2 = r_2 = 0.5, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, t = -3, z = 1$. **d**–**f** are the 2-D contours for **a**-**c** in xy-plane

where $P_1 = p_1x + q_1y + r_1z$, $P_2 = p_2x + q_2y + r_2z$, $Q_1 = \alpha_3 p_1^2 + p_1^3 q_1 + \alpha_4 r_1^2$, and $Q_2 = \alpha_3 p_2^2 + p_2^3 q_2 + \alpha_4 r_2^2$.

4.3 Three-dispersive-soliton solution

A three-dispersive-soliton solution can be obtained by assuming function f as

$$f(x, y, z, t) = 1 + e^{\Phi_1} + e^{\Phi_2} + e^{\Phi_3} + e^{\Phi_1+\Phi_2} + e^{\Phi_1+\Phi_3} + e^{\Phi_2+\Phi_3} + e^{\Phi_1+\Phi_2+\Phi_3}, \tag{38}$$

Therefore, we can get

$$f_x = \sum_{k=1}^{3} p_k e^{\Phi_k} + \sum_{1\le m<n\le 3} (p_m + p_n) e^{\Phi_m+\Phi_n} + \left(\sum_{k=1}^{3} p_k\right) e^{\Phi_1+\Phi_2+\Phi_3}, \tag{39}$$

$$f_{xx} = \sum_{k=1}^{3} p_k^2 e^{\Phi_k} + \sum_{1\le m<n\le 3} (p_m + p_n)^2 e^{\Phi_m+\Phi_n} + \left(\sum_{k=1}^{3} p_k\right)^2 e^{\Phi_1+\Phi_2+\Phi_3}. \tag{40}$$

By substituting Eqs. (38), (39) and (40) into Eq. (27), we obtain the three-dispersive-soliton solution.

5 Results and findings

The complete integrability of the proposed generalized nonlinear evolution Eq. (1) can generate various solutions including kinks, breathers, lumps and others. Using proper selection of parameter, we established the rogue waves of higher orders with center-parameter (β, γ) doing computation symbolically and demonstrated the dynamical graphics for the solutions. Also, we showed the dispersive-soliton solutions with appro-

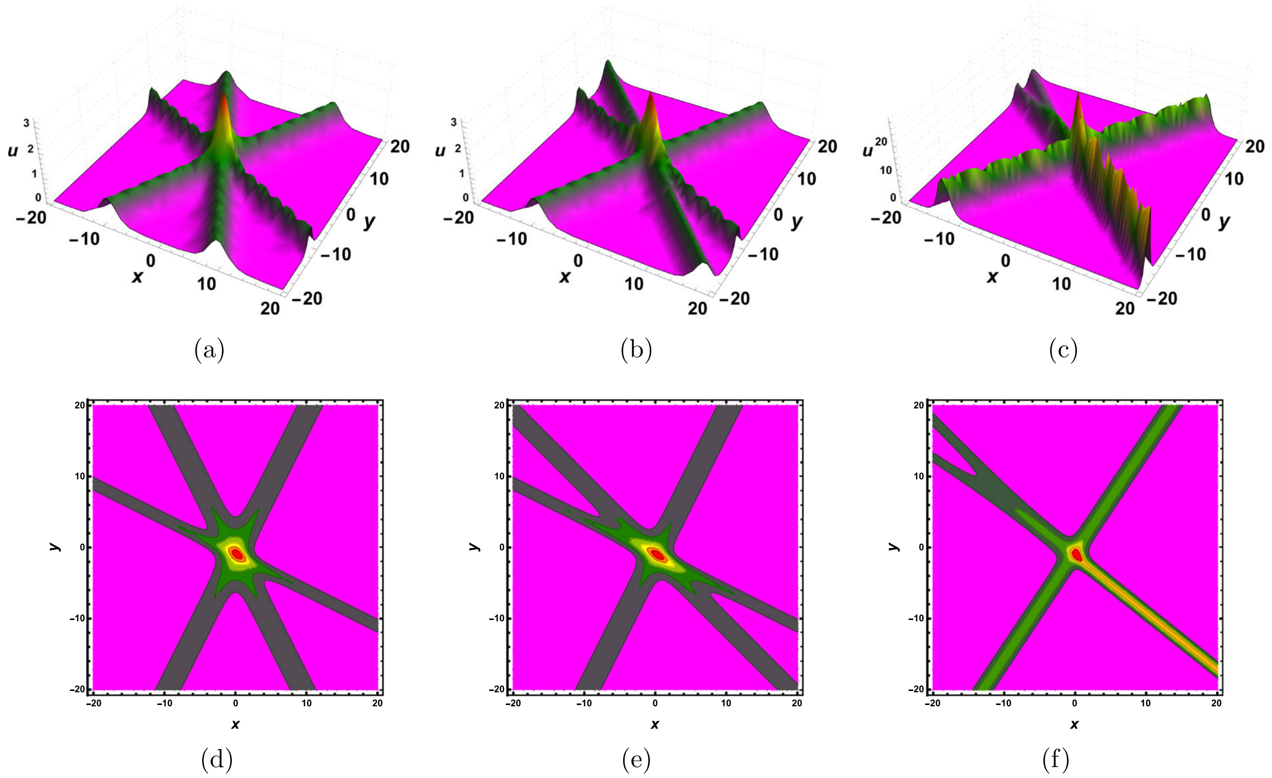

Fig. 6 Interaction of three dispersive solitons with the auxiliary function (38) having values: **a** $p_1 = 1, q_1 = r_1 = 0.5, p_2 = -1, q_2 = r_2 = 0.5, p_3 = 1, q_3 = r_3 = 2, \alpha_1 = 1, \alpha_2 = 3, \alpha_3 = \alpha_4 = 1, t = 0, z = 1$; **b** $p_1 = q_1 = r_1 = 1, p_2 = -1, q_2 = r_2 = 0.5, p_3 = 1, q_3 = r_3 = 2, \alpha_1 = 1, \alpha_2 = 3, \alpha_3 = \alpha_4 = 1, t = 0, z = 1$; and **c** $p_1 = 1, q_1 = 1.5, r_1 = 1, p_2 = -1.5, q_2 = 1, r_2 = 0.5, p_3 = q_3 = 1, r_3 = 2, \alpha_1 = 2, \alpha_2 = 0.5, \alpha_3 = \alpha_4 = 1, t = 0, z = 1$. **d**–**f** are the 2-D contours for **a**-**c** in xy-plane

priate constant parameters in the dispersion relation. In this context, the finding explanations are as follows:

- In Fig. 1, we illustrate first-order rogue-wave solutions having singularity about center-parameter $U = \beta$ with the values: $\alpha_1 = -10, \alpha_2 = \alpha_3 = \alpha_4 = 1$, and center-parameter: $(\beta, \gamma) = (0, 0), (-1, 0)$ and $(2, 1)$ for (a)-(c).
- Figure 2 showcases the second-order rogue waves that depicts that two rogue waves having singularity depend on center parameters and occurs in a line with the constant values: $\alpha_1 = 10, \alpha_2 = -10, \alpha_3 = 35, \alpha_4 = 2, \beta = 0, \gamma = 0$; $\alpha_1 = 5, \alpha_2 = -0.5, \alpha_3 = \alpha_4 = 1, \beta = 1, \gamma = 7$; and $\alpha_1 = 1, \alpha_2 = -3, \alpha_3 = \alpha_4 = 0.01, \beta = 0, \gamma = 1$ for (a)-(c).
- In Fig.3, graphics depicts the third-order rogue waves having singularities about center-parameter (β, γ). Four rogue waves occur in a circular path with the constant values: $\alpha_1 = -0.1, \alpha_2 = -1, \alpha_3 = 0.5, \alpha_4 = 0.5$, and center-parameter: $\beta = \gamma = 0$; $\beta = \gamma = 15$; and $\beta = 15, \gamma = -20$ w.r.t. (a)-(c).
- Figure 4 illustrates the dispersive solitons in (a) and (c) are moving in positive direction, while (b) is moving in negative direction, justifying by the graphics (g)-(i) with the values $t = 0, t = 2, t = 4, t = 6$. Single solitons exist for singularity with the values: $p_1 = q_1 = r_1 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, t = 0, z = 1$; $p_1 = 1, q_1 = -0.3, r_1 = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, t = 0, z = 1$ and $p_1 = 1, q_1 = -0.4, r_1 = 0, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, t = 0, z = 1$ for (a)-(c).
- In Fig. 5, we showcase the interactions of two dispersive solitons, where all interactions show a X-shape interaction with the values: $p_1 = 1, q_1 = r_1 == 0.5, p_2 = -1, q_2 = r_2 = 0.5, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, t = 0, z = 1$; $p_1 = 1, q_1 = 0.7, r_1 = 0.5, p_2 = -1, q_2 = r_2 =$

$0.5, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, t = 3, z = 1$; and $p_1 = 1, q_1 = 0.7, r_1 == 0.5$, $p_2 = -1, q_2 = r_2 = 0.5, \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1, t = -3, z = 1$ for (a)-(c).

- Figure 6 depicts the interaction of three dispersive solitons, where figures (a) and (b) form an X-shape interactions, while figure (c) shows the Y-shape interaction with the values: $p_1 = 1, q_1 = r_1 = 0.5$, $p_2 = -1, q_2 = r_2 = 0.5$, $p_3 = 1, q_3 = r_3 = 2, \alpha_1 = 1, \alpha_2 = 3, \alpha_3 = \alpha_4 = 1, t = 0, z = 1$; $p_1 = q_1 = r_1 = 1, p_2 = -1, q_2 = r_2 = 0.5$, $p_3 = 1, q_3 = r_3 = 2, \alpha_1 = 1, \alpha_2 = 3, \alpha_3 = \alpha_4 = 1, t = 0, z = 1$; and $p_1 = 1, q_1 = 1.5, r_1 = 1$, $p_2 = -1.5, q_2 = 1, r_2 = 0.5$, $p_3 = q_3 = 1, r_3 = 2, \alpha_1 = 2, \alpha_2 = 0.5, \alpha_3 = \alpha_4 = 1, t = 0, z = 1$ for (a)-(c).

6 Conclusions

In conclusion, this article explored a new generalized (3+1)-D Painlevé-type nonlinear evolution equation. It analyzed the Painlevé test to check the complete integrability of the proposed equation. It obtained trilinear equations in auxiliary functions using Cole–Hopf transformations or logarithmic transformations. It generated the higher-order solutions for rogue wave with center-parameter (β, γ) and dispersive-soliton solutions via the symbolic computation approach and dispersive-soliton assumption, respectively. This work established the rogue waves and dispersive solitons up to the third order by choosing selective values for different parameters and discussing their results and findings. We used symbolic computer algebra system software *Mathematica* to generate the dynamics for the solutions of higher-order rogue wave with several center parameters and dispersive solitons with parameters present in the dispersion relation.

The proposed equation is a generalized equation with applications in soliton theory and nonlinear waves. Thus, the proposed equation is having a future scope to study different water-waves including kinks, breathers, lumps, and others solutions. As we have utilized the methodologies the direct symbolic approach to create the rogue waves and dispersive-soliton assumption for dispersive solitons, researchers and engineers can examine the proposed equation with other strategies and approaches such as the symbol calculation method based on neural networks proposed by Zhang et al., Lie symmetry analysis, Darboux transformation, Hirota's bilinearization method and others concerned in the introduction section.

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Data availability This study has not made use of any data from other sources.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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ABSTRACT

This work investigates a (2 + 1)-dimensional shallow water wave equation of ion-acoustic waves in plasma physics. It comprehensively analyzes Cole–Hopf transformations concerning dimensions x, y, and t and obtains the dispersion for a phase variable of this equation. We show that the soliton solutions are independent of the different logarithmic transformations for the investigated equation. We also explore the linear equations in the auxiliary function f present in Cole–Hopf transformations. We study this equation's first- and second-order rogue waves using a generalized N-rogue wave expression from the N-soliton Hirota technique. We generate the rogue waves by applying a symbolic technique with β and γ as center parameters. We create rogue wave solutions for first- and second-order using direct computation for appropriate choices of several constants in the equation and center parameters. We obtain a trilinear equation by transforming variables ξ and y via logarithmic transformation for u in the function F. We harness the computational power of the symbolic tool *Mathematica* to demonstrate the graphics of the soliton and center-controlled rogue wave solutions with suitable choices of parameters. The outcomes of this study transcend the confines of plasma physics, shedding light on the interaction dynamics of ion-acoustic solitons in three-dimensional space. The equation's implications resonate across diverse scientific domains, encompassing classical shallow water theory, fluid dynamics, optical fibers, nonlinear dynamics, and many other nonlinear fields.

I. INTRODUCTION

Shallow water waves (SWWs) are an exquisite phenomenon described by waves propagating at depths significantly smaller than their wavelength.[1–6] These waves display distinguishing behaviors due to the influence of the sea or lake floor. It makes them a subject of fascination for scientists, mathematicians, and physicists. SWWs often find their origin in coastal regions with reasonably shallow water depths. Various characteristics, including wind, tides, and seismic activity, can cause them. The interaction between the wind and the water surface is a primary driver for creating interesting waves that gracefully transit the shallows. The remarkable characteristics of shallow water waves make them relevant in various applications. Coastal engineering leverages the acquaintance of SWW to design structures that can withstand wave action.

Furthermore, they play a vital role in activities such as surfing, where enthusiasts harness the energy of these waves for recreational purposes. Their presence not only enchants coastal landscapes but also serves as a canvas for scientific exploration and practical applications in engineering and recreation. From a scientific perspective, studying SWWs provides valuable insight into fluid dynamics and the complex interactions between water and its surroundings. Comprehending these waves is integral to forecasting coastal erosion, managing water resources, and comprehending the broader implications of climate change on coastal ecosystems.

Solitons, or solitary and self-sustaining waves,[7–15] are unprecedented phenomena in wave dynamics. Unlike conventional waves, solitons maintain their structure and energy, traveling undisturbed long distances without dissipating or losing their form. The source of solitons can be diverse, emerging from nonlinear interactions in diverse mediums such as ocean, optical fibers, and plasma. Notable for their ability to resist dispersion and sustain stability, solitons often emerge due to a delicate balance between nonlinearity and dispersion. Solitons find applications in a range of scientific and technological fields. They serve as data carriers, ensuring signal integrity over vast distances in optical communication. Their presence in fluid dynamics contributes to understanding rogue waves, and in plasma physics, they play a crucial role in sustaining plasma stability. Solitons challenge traditional wave theories, offering insight into nonlinear phenomena and the preservation of wave coherence. Their occurrence in various natural systems extends our understanding of complex wave interactions. Therefore, from revolutionizing communication technologies to unraveling mysteries in fluid dynamics, solitons stand as silent yet powerful contributors to the tapestry of scientific exploration.

Rogue waves, often called giant solitary waves emerging unexpectedly in the ocean's vastness, manifest as localized phenomena in space and time, boasting a considerable amplitude. These enigmatic occurrences, documented in sources, such as Refs. 16–23, are unpredictable and ubiquitous, posing potential hazards to individuals. Rogue waves materialize randomly as tiny waves converge the energy in a confined territory. A crucial application of this research lies in enhancing marine safety. By developing models and prediction algorithms, scientists aim to provide early detection and warning systems to prevent accidents triggered by rogue waves. This knowledge significantly benefits maritime, offshore petroleum platforms, and seaside infrastructure industries. Consequently, a comprehensive understanding of the dynamics of rogue waves can lead to improved operational safety and cost-effective solutions, enabling the construction of safer structures and formulating strategies to mitigate their impact. Moreover, delving into the causes and dynamics of rogue waves contributes to our expanding comprehension of complicated systems, the interactions of nonlinear waves, and the emergence of severe phenomena across various fields of physics and mathematics. The study of rogue waves transcends maritime concerns, offering valuable insight into broader scientific principles and the intricacies of nonlinear wave behavior.

Researchers and scientists have studied the nonlinear partial differential equations (PDEs) or nonlinear evolution equations[24–28] using several methods for obtaining the exact solutions, such as the inverse scattering method,[29,30] the Darboux transformation,[31–33] the simplified Hirota's technique,[34–36] bilinear neural/residual network method,[37–43] Hirota's bilinear method,[44–47] the Lie symmetry analysis,[48–50] the Bäcklund transformation,[51–53] and other techniques.

This research investigates the Cole–Hopf transformations concerning different dimensions and rogue waves for a $(2+1)$-dimensional SWW equation[54–56]

$$u_{tt} - u_{xx} - u_{yy} + u_x u_{xt} + u_y u_{yt} - u_{xxtt} - u_{yytt} = 0. \tag{1}$$

In 1978, Yajima *et al.*[54] modeled this equation in three-dimensional interactions of ion-acoustic solitons in collisionless plasmas. They studied it using Hirota's bilinear method and discussed the one- and two-soliton solutions for the same. In the continuation, Kako and Yajima[55] studied this model of ion-acoustic solitons in collisionless plasmas in two-dimensional space. They showed the dynamics for the interaction of two soliton solutions for the obtained solution with some appropriate parameters. Also, they showed the interaction of two sinusoidal waves dynamically with chosen constants. In 1994, Clarkson and Mansfield[56] quoted this Eq. (1) in their work on a SWW equation, in which they studied a generalized SWW equation by non-Painlevé behavior and dynamically showed the solitons' interaction solutions and breathers using Lie symmetry analysis for classical and non-classical symmetries. We found it fascinating that no more work has been done in the literature on this equation as far as we know, whereas this has an exciting pattern for partial derivatives in its linear terms. This gave us an idea to think about several Cole–Hopf transformations in different dimensions.

In the structure of the manuscript, Sec. II analyzes the Cole–Hopf transformations concerning dimensions x, y, and t and obtains the dispersion for a phase variable of the investigated equation with a discussion of soliton solutions for the different transformations. Section III studies the equation's first- and second-order solutions of rogue waves using a generalized N-rogue wave expression from the N-soliton Hirota technique with center parameters. We compute a trilinear equation in an auxiliary function using the logarithmic transformation and create rogue wave solutions up to second order for suitable values of center parameters and several constants in the equation using a direct computation technique. Section IV discusses the obtained solutions and the dynamical analysis. Section V, in the end, concludes our findings and future scope.

II. ANALYSIS OF COLE–HOPF TRANSFORMATIONS

The Cole–Hopf transformation is a mathematical technique used in partial differential equations (PDEs), particularly in studying certain nonlinear PDEs. Cole and Hopf[57,58] created the transformation in the 1950s to simplify and sometimes linearize certain types of nonlinear PDEs. It is most commonly associated with the Korteweg–de Vries (KdV) equation, a nonlinear PDE that involves nonlinear and dispersive terms and describes the propagation of long, weakly nonlinear waves, such as water waves in shallow canals. The Cole–Hopf transformation has been a valuable tool in the study of soliton theory and integrable systems, where it allows researchers to comprehend the manners of certain nonlinear wave equations and uncover essential properties, such as the existence of localized solutions, solitary waves, and several other solutions that can persist in specific nonlinear systems. The Cole–Hopf transformation, in general, is given as

$$u = R(\ln f)_{x^p}, \tag{2}$$

for a given nonlinear PDE, where p represents the order of partial derivative concerning x leaning on the balance of the higher-order and nonlinear terms in the PDE.

In order to create the said transformation, we need to get the dispersion with the help of the phase variable. We consider the phase variable as

$$\alpha_i = p_i x + q_i y - w_i t, \tag{3}$$

where w_i and $p_i, q_i; i \in N$ represent dispersion and constants respectively. On substituting

$$u = e^{\alpha_i} \tag{4}$$

into the linear terms of Eq. (1), we get the w_i as

$$w_i = \pm \frac{\sqrt{p_i^2 + q_i^2}}{\sqrt{1 - (p_i^2 + q_i^2)}} \tag{5}$$

with $p_i^2 + q_i^2 < 1$ for getting real valued dispersion.

Now, we assume the three Cole–Hopf transformations in different dimensions x, y, and t known as spatial and temporal coordinates as

$$u = u_1 = R_1(\ln f)_x, \tag{6}$$

$$u = u_2 = R_2(\ln f)_y, \tag{7}$$

$$u = u_3 = R_3(\ln f)_t, \tag{8}$$

where $R_i; i = 1, 2, 3$ are non-zero constants and $f = f(x, y, t)$ is an auxiliary function, which will be determined later. To determine the value of $R_i; i = 1, 2, 3$ in Eqs. (6)–(8), we consider

$$f(x, y, t) = 1 + e^{\alpha_i} = 1 + e^{p_i x + q_i y - w_i t}. \tag{9}$$

On substituting Eqs. (6), (7), or (8) with Eq. (9) into Eq. (1), we get the solution for R_i as

$$R_1 = \frac{12\sqrt{p_1^2 + q_1^2}}{p_1\sqrt{1 - (p_1^2 + q_1^2)}}, \quad R_2 = \frac{12\sqrt{p_1^2 + q_1^2}}{q_1\sqrt{1 - (p_1^2 + q_1^2)}}, \quad R_3 = -12, \tag{10}$$

for Eqs. (6), (7), and (8), respectively. Thus, the transformations will be as

$$\begin{aligned} u_1 &= \frac{12\sqrt{p_1^2 + q_1^2}}{p_1\sqrt{1 - (p_1^2 + q_1^2)}}(\ln f)_x, \\ u_2 &= \frac{12\sqrt{p_1^2 + q_1^2}}{q_1\sqrt{1 - (p_1^2 + q_1^2)}}(\ln f)_y, \\ u_3 &= -12(\ln f)_t. \end{aligned} \tag{11}$$

On utilizing the above transformations Eq. (11) into Eq. (1), we obtain transformed equations for Eq. (1) in auxiliary function f as

$$\begin{aligned} & f_{xtt}f^4 - f_{xxx}f^4 - f_{xxxtt}f^4 - f_{xyy}f^4 - f_{xyytt}f^4 - f_{tt}f_xf^3 - 2f_tf_{xt}f^3 + 3f_xf_{xx}f^3 + 3f_{xtt}f_{xx}f^3 + 6f_{xt}f_{xxt}f^3 \\ & + R_1f_{xx}f_{xxt}f^3 + 3f_xf_{xxtt}f^3 + f_{tt}f_{xxx}f^3 + 2f_tf_{xxxt}f^3 + R_1f_{xy}f_{xyt}f^3 + f_{tt}f_{xyy}f^3 + 2f_tf_{xyyt}f^3 + 2f_{xy}f_yf^3 + 2f_{xytt}f_yf^3 \\ & + 4f_{xyt}f_{yt}f^3 + 2f_{xy}f_{ytt}f^3 + f_xf_{yy}f^3 + f_{xtt}f_{yy}f^3 + 2f_{xt}f_{yyt}f^3 + f_xf_{yytt}f^3 - 2f_x^3f^2 - 12f_xf_{xt}^2f^2 - R_1f_tf_{xx}^2f^2 - R_1f_tf_{xy}^2f^2 \\ & - 2f_xf_y^2f^2 - 2f_{xtt}f_y^2f^2 - 4f_xf_{yt}^2f^2 + 2f_t^2f_xf^2 - 6f_x^2f_{xtt}f^2 - 6f_{tt}f_xf_{xx}f^2 - 12f_tf_{xt}f_{xx}f^2 - 2R_1f_xf_{xt}f_{xx}f^2 - R_1f_x^2f_{xxt}f^2 \\ & - 12f_tf_xf_{xxt}f^2 - 2f_t^2f_{xxx}f^2 - 2f_t^2f_{xyy}f^2 - 4f_{tt}f_{xy}f_yf^2 - R_1f_{xt}f_{xy}f_yf^2 - 8f_tf_{xyt}f_yf^2 - R_1f_xf_{xyt}f_yf^2 - 8f_tf_{xy}f_{yt}f^2 \\ & - R_1f_xf_{xy}f_{yt}f^2 - 8f_{xt}f_yf_{yt}f^2 - 4f_xf_yf_{ytt}f^2 - 2f_{tt}f_xf_{yy}f^2 - 4f_tf_{xt}f_{yy}f^2 - 4f_tf_xf_{yyt}f^2 + 6f_{tt}f_x^3f + 6f_{tt}f_xf_y^2f \\ & + 12f_tf_{xt}f_y^2f + R_1f_xf_{xt}f_y^2f + 2R_1f_x^3f_{xt}f + 36f_tf_x^2f_{xt}f + 3R_1f_tf_x^2f_{xx}f + 18f_t^2f_xf_{xx}f + 12f_t^2f_{xy}f_yf + 3R_1f_tf_xf_{xy}f_yf \\ & + R_1f_x^2f_yf_{yt}f + 24f_tf_xf_yf_{yt}f + 6f_t^2f_xf_{yy}f - 2R_1f_tf_x^4 - 24f_t^2f_x^3 - 2R_1f_tf_x^2f_y^2 - 24f_t^2f_xf_y^2 = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} & -f_{xxytt}f^4 + f_{ytt}f^4 - f_{yyy}f^4 - f_{yyytt}f^4 + f_{tt}f_{xxy}f^3 + 2f_tf_{xxyt}f^3 + 2f_xf_{xy}f^3 + 2f_{xtt}f_{xy}f^3 + 4f_{xt}f_{xyt}f^3 + R_2f_{xy}f_{xyt}f^3 \\ & + 2f_xf_{xytt}f^3 - f_{tt}f_yf^3 + f_{xx}f_yf^3 + f_{xxtt}f_yf^3 - 2f_tf_{yt}f^3 + 2f_{xxt}f_{yt}f^3 + f_{xx}f_{ytt}f^3 + 3f_yf_{yy}f^3 + 3f_{ytt}f_{yy}f^3 + 6f_{yt}f_{yyt}f^3 \\ & + R_2f_{yy}f_{yyt}f^3 + 3f_yf_{yytt}f^3 + f_{tt}f_{yyy}f^3 + 2f_tf_{yyyt}f^3 - 2f_y^3f^2 - R_2f_tf_{xy}^2f^2 - 12f_yf_{yt}^2f^2 - R_2f_tf_{yy}^2f^2 - 2f_t^2f_{xxy}f^2 \\ & - 4f_{tt}f_xf_{xy}f^2 - 8f_tf_{xt}f_{xy}f^2 - 8f_tf_xf_{xyt}f^2 + 2f_t^2f_yf^2 - 2f_x^2f_yf^2 - 4f_{xt}^2f_yf^2 - 4f_xf_{xtt}f_yf^2 - 2f_{tt}f_{xx}f_yf^2 - 4f_tf_{xxt}f_yf^2 \\ & - R_2f_{xt}f_{xy}f_yf^2 - R_2f_xf_{xyt}f_yf^2 - 8f_xf_{xt}f_{yt}f^2 - 4f_tf_{xx}f_{yt}f^2 - R_2f_xf_{xy}f_{yt}f^2 - 2f_x^2f_{ytt}f^2 - 6f_y^2f_{ytt}f^2 - 6f_{tt}f_yf_{yy}f^2 \\ & - 12f_tf_{yt}f_{yy}f^2 - 2Rf_yf_{yt}f_{yy}f^2 - Rf_y^2f_{yyt}f^2 - 12f_tf_yf_{yyt}f^2 - 2f_t^2f_{yyy}f^2 + 6f_{tt}f_y^3f + Rf_xf_{xt}f_y^2f \\ & + 12f_t^2f_xf_{xy}f + 6f_{tt}f_x^2f_yf + 24f_tf_xf_{xt}f_yf + 6f_t^2f_{xx}f_yf + 3Rf_tf_xf_{xy}f_yf + 2R_2f_y^3f_{yt}f + 12f_tf_x^2f_{yt}f + 36f_tf_y^2f_{yt}f \\ & + R_2f_x^2f_yf_{yt}f + 3R_2f_tf_y^2f_{yy}f + 18f_t^2f_yf_{yy}f - 2R_2f_tf_y^4 - 24f_t^2f_y^3 - 2R_2f_tf_x^2f_y^2 - f^4f_{xxy} - 24f_t^2f_x^2f_y = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} & f^2f_{tt} - f^2f_{xx} - f^2f_{xxtt} - f^2f_{yy} - f^2f_{yytt} + 4f_tf_xf_{xt} - 2f_t^2f_{xx} \\ & + 2ff_tf_{xxt} + 4f_tf_yf_{yt} - 2f_t^2f_{yy} + 2ff_tf_{yyt} - ff_t^2 - 2f_{tt}f_x^2 \\ & + ff_{tt}f_{xx} - 2f_{tt}f_y^2 + ff_{tt}f_{yy} + 2ff_xf_{xtt} + ff_x^2 - 4ff_{xt}^2 \\ & + 2ff_yf_{ytt} + ff_y^2 - 4ff_{yt}^2 = 0, \end{aligned} \tag{14}$$

for Eqs. (6), (7), and (8), respectively.

Considering the function f in any of the above-mentioned equation as

$$f = 1 + e^{\alpha_1} = 1 + e^{p_1x + q_1y - \frac{\sqrt{p_1^2+q_1^2}}{\sqrt{1-(p_1^2+q_1^2)}}t}. \tag{15}$$

Thus, by putting up the expression for f from Eq. (15) into any transformation of u in Eq. (11), we get the same solution as

$$u = \frac{12\sqrt{p_1^2 + q_1^2}e^{p_1x+q_1y}}{\sqrt{1 - (p_1^2 + q_1^2)}\left(e^{\frac{t\sqrt{p_1^2-q_1^2}}{\sqrt{1-(p_1^2+q_1^2)}}} + e^{p_1x+q_1y}\right)}, \tag{16}$$

which shows that the $(2+1)$-dimensional SWW equation (1) is independent of the Cole–Hopf transformations and any transformation can be used to get the soliton solutions. Readers can follow the works[54,55] to dive into the soliton solutions and their interactions.

III. CENTER-CONTROLLED ROGUE WAVES

On transforming $u = u(\xi, y)$ with $\xi = x - h * t$, Eq. (1), we get

$$h^2(u_{\xi\xi} - u_{\xi\xi\xi\xi} - u_{\xi\xi yy}) - h(u_\xi u_{\xi\xi} + u_{\xi y}u_y) - u_{\xi\xi} - u_{yy} = 0. \quad (17)$$

Considering the phase Φ_i in Eq. (17) as

$$\Phi_i = p_i\xi - w_i y, \quad (18)$$

where p_i and w_i; $i \in N$ are parameters and dispersion, respectively. By putting $u(\xi, y) = e^{\Phi_i}$ in Eq. (17) for linear terms, we obtain

$$w_i = \pm\frac{\sqrt{-h^2p_i^4 + h^2p_i^2 - p_i^2}}{\sqrt{h^2p_i^2 + 1}}. \quad (19)$$

We take the dependent variable transformation as

$$u(\xi, y) = R(\log F)_\xi, \quad (20)$$

and put it in Eq. (17) with Eq. (19) and $F = 1 + e^{\Phi_1}$, then we get R as

$$R = 12h.$$

So, the Eq. (20) becomes

$$u(\xi, y) = u_0 + 12h(lnF)_\xi, \quad (21)$$

where u_0 is a constant parameter. On substituting Eq. (21) into Eq. (17), we get a trilinear equation in $F(\xi, y)$ as

$$\begin{aligned}&F^2h^2F_{\xi\xi\xi\xi} - F^2h^2F_{\xi\xi} + F^2h^2F_{\xi\xi yy} + F^2F_{\xi\xi} + F^2F_{yy}\\&- 4Fh^2F_\xi F_{\xi\xi\xi} - 2Fh^2F_\xi F_{\xi yy} + Fh^2F_\xi^2 + 3Fh^2F_{\xi\xi}^2\\&+ 4Fh^2F_{\xi y}^2 - 4h^2F_\xi F_{\xi y}F_y + 2h^2F_{\xi\xi}F_y^2 - 2Fh^2F_{\xi\xi y}F_y\\&+ 2h^2F_\xi^2F_{yy} - Fh^2F_{\xi\xi}F_{yy} - FF_\xi^2 - FF_y^2 = 0. \end{aligned} \quad (22)$$

In 2023, Kumar and Mohan[59] generalized a direct technique to construct N-order rogue waves using N-soliton solution in Hirota's technique which was first discussed by Yang *et al.*[60] in 2022, for a $(3+1)$-dimensional KdV–BBM equation with limit technique of long wave. It gives the generalized representation of N-rogue waves as

$$F_N(\xi, y) = \sum_{j=0}^{\frac{N^2+N}{2}}\sum_{k=0}^{j} s_{N^2+N-2j,2k}(y)^{2k}(\xi)^{N^2+N-2j}, \quad (23)$$

which has a resemblance to the functions used in symbolic computational approach[61,62] by Zhaqilao.[63] This can be expressed with center-controlled parameter as

$$\begin{aligned}F(\xi, y) &= \widehat{F_N}(\xi, y, \beta, \gamma)\\&= \sum_{j=0}^{\frac{N^2+N}{2}}\sum_{k=0}^{j} s_{N^2+N-2j,2k}(y-\gamma)^{2k}(\xi-\beta)^{N^2+N-2j},\end{aligned} \quad (24)$$

where $s_{i,j}$; $i, j \in \{0, 2, \ldots, j(j+1)\}$ are constants and (β, γ) are center parameters.

A. First-order solution of rogue waves

Considering $F(\xi, y)$ with $N = 1$ in Eq. (24) as

$$F(\xi, y) = s_{2,0}\xi^2 + s_{0,2}y^2 + s_{0,0}, \quad (25)$$

and substituting it into Eq. (22) gives a system by equating the coefficients for distinct powers of $\xi^m y^n$; $m, n \in Z$ to zero as

$$\begin{aligned}&2h^2s_{2,0}^3 - 2s_{2,0}^3 + 2s_{0,2}s_{2,0}^2 = 0,\\&12h^2s_{2,0}^3 + 12h^2s_{0,2}s_{2,0}^2 + 4s_{0,0}s_{0,2}s_{2,0} = 0,\\&12h^2s_{0,2}s_{2,0}^2 + 12h^2s_{0,2}^2s_{2,0} - 4h^2s_{0,0}s_{0,2}s_{2,0} + 4s_{0,0}s_{0,2}s_{2,0} = 0.\end{aligned} \quad (26)$$

Solving above system gives constants as

$$s_{0,0} = \frac{3h^2(h^2-2)s_{2,0}}{1-h^2}, \quad s_{0,2} = (1-h^2)s_{2,0}, \quad s_{2,0} = s_{2,0}. \quad (27)$$

Thus, Eq. (25) with Eq. (27) will be as

$$\begin{aligned}F(\xi, y) &= \widehat{f_1}(\xi, y, \beta, \gamma)\\&= s_{2,0}\left((\beta-\xi)^2 + (1-h^2)(y-\gamma)^2 + \frac{3(h^2-2)h^2}{1-h^2}\right),\end{aligned} \quad (28)$$

which gives a solution of Eq. (22). We get a first-order solution of rogue waves on substituting Eq. (28) into Eq. (21) as

$$u(\xi, y) = u_0 + \frac{24h(\xi-\beta)}{(\beta-\xi)^2 + (1-h^2)(y-\gamma)^2 + \frac{3(h^2-2)h^2}{1-h^2}}. \quad (29)$$

B. Second-order solution of rogue waves

Taking auxiliary function $F(\xi, y)$ with $N = 2$ in Eq. (24) as

$$\begin{aligned}F(\xi, y) &= s_{6,0}\xi^6 + s_{4,0}\xi^4 + s_{2,0}\xi^2 + s_{0,6}y^6 + s_{2,4}\xi^2y^4 + s_{0,4}y^4\\&+ s_{4,2}\xi^4y^2 + s_{2,2}\xi^2y^2 + s_{0,2}y^2 + s_{0,0},\end{aligned} \quad (30)$$

and put it in trilinear Eq. (22). On equating zero the coefficients for distinct powers of $\xi^m y^n$; $m, n \in Z$, we obtain a system, which gives the constant values as

$$\begin{aligned}&s_{0,0} = \frac{h^6(286h^6 - 1383h^4 + 2548h^2 - 2076)s_{4,2}}{(h^2-1)^4},\\&s_{0,2} = \frac{h^4(144h^4 - 627h^2 + 958)s_{4,2}}{3(h^2-1)^2},\\&s_{0,4} = \frac{1}{3}h^2(33h^2-50)s_{4,2}, \quad s_{0,6} = \frac{1}{3}(h^2-1)^2s_{4,2},\\&s_{2,0} = -\frac{h^4(144h^4 - 507h^2 + 238)s_{4,2}}{3(h^2-1)^3},\\&s_{2,2} = -\frac{6h^2(h^2-6)s_{4,2}}{h^2-1}, \quad s_{2,4} = -(h^2-1)s_{4,2},\\&s_{4,0} = \frac{h^2(33h^2-58)s_{4,2}}{3(h^2-1)^2}, \quad s_{4,2} = s_{4,2}, \quad s_{6,0} = \frac{s_{4,2}}{3(1-h^2)}.\end{aligned} \quad (31)$$

So, the Eq. (25) with Eq. (31) becomes

$$
\begin{aligned}
F(\xi, y) &= \widehat{f}_2(\xi, y, \beta, \gamma) \\
&= \frac{s_{4,2}}{3}\Bigg(\frac{(33h^2-58)h^2(\beta-\xi)^4}{(h^2-1)^2} - \frac{(\beta-\xi)^6}{h^2-1} \\
&\quad - \frac{18(h^2-6)h^2(\beta-\xi)^2(y-\gamma)^2}{h^2-1} \\
&\quad - 3(h^2-1)(\beta-\xi)^2(y-\gamma)^4 + (33h^2-50)h^2(y-\gamma)^4 \\
&\quad + (h^2-1)^2(y-\gamma)^6 - \frac{(144h^4-507h^2+238)h^4(\beta-\xi)^2}{(h^2-1)^3} \\
&\quad + \frac{(144h^4-627h^2+958)h^4(y-\gamma)^2}{(h^2-1)^2} \\
&\quad + \frac{3(286h^6-1383h^4+2548h^2-2076)h^6}{(h^2-1)^4} \\
&\quad + 3(\beta-\xi)^4(y-\gamma)^2\Bigg),
\end{aligned} \tag{32}
$$

which gives a solution of Eq. (22). We get a second-order solution of rogue waves on substituting Eq. (32) into Eq. (21) as

$$
u(\xi, y) = u_0 + 12h(\ln \widehat{f}_2(\xi, y, \beta, \gamma))_\xi. \tag{33}
$$

IV. RESULTS AND DISCUSSION

Our investigation shows that the $(2+1)$-dimensional SWW equation governing ion-acoustic waves in plasma physics can have different Cole–Hopf transformations in different dimensions x, y, and t. The analysis of these transformations showed that the soliton solutions for this SWW equation are independent of the Cole–Hopf transformations and give the same solution as discussed in Sec. II. By selecting the appropriate parameters, we found the first- and second-order solutions of rogue waves with center parameters (β, γ) with the said symbolic approach and dynamically showed the graphics of the obtained solutions. The exploration of the evolutionary processes behind rogue waves is imperative, capturing the attention of numerous academics. Their excessively steep height sets rogue waves apart, sometimes surpassing the magnitude of neighboring waves. These unique characteristic challenges traditional linear wave models, prompting a focus on nonlinear wave dynamics in understanding the mechanics and predicting the occurrence of these formidable waves. Therefore, we explain the results and findings as follows:

- In Fig. 1, we illustrate the solitons for the solution Eq. (16) with Eq. (15) with respect to the singularity about the x axis. (a)–(c) show the dynamics of solitons with values (a) $p_1 = 0.8$, $q_1 = 0.3$, (b) $p_1 = 0.7$, $q_1 = -0.7$, and (c) $p_1 = -0.8$, $q_1 = 0.3$.
- Figure 2 depicts the first-order solution of rogue waves with center parameters (β, γ). It shows single rogue waves concerning singularity through center parameters (β, γ) with values (a)

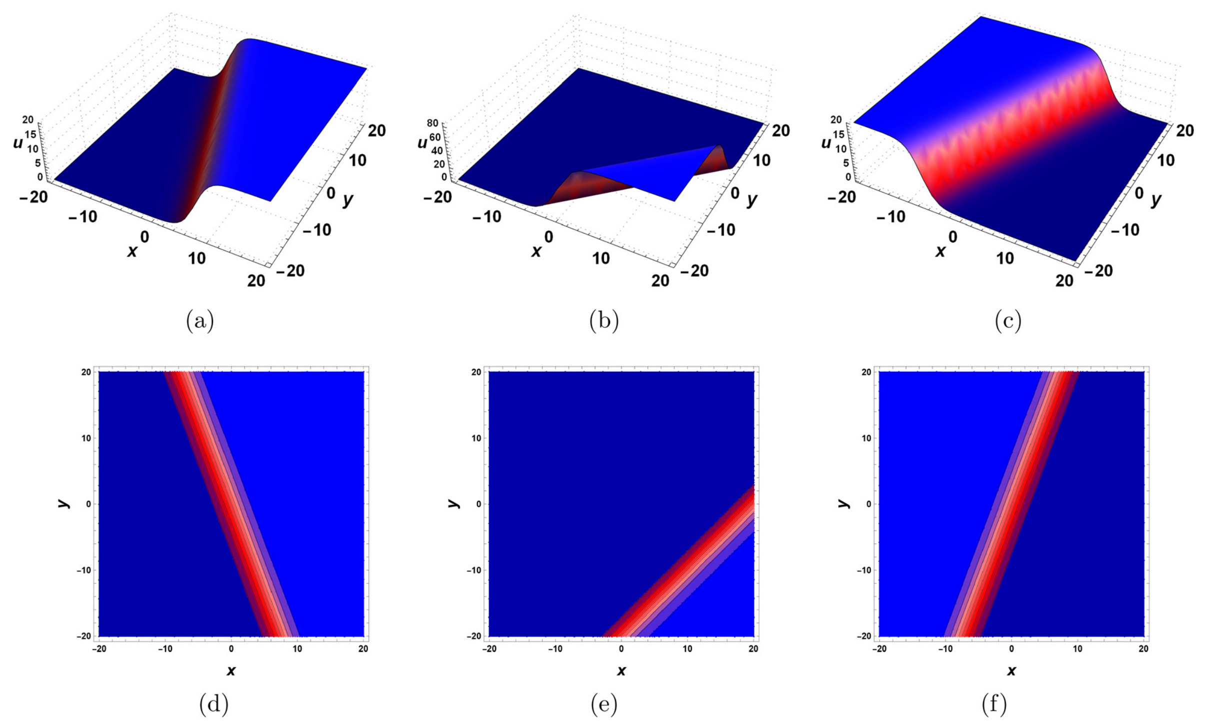

FIG. 1. Solitons for the solution (16) with (15) having values (a) $p_1 = 0.8$, $q_1 = 0.3$, (b) $p_1 = 0.7$, $q_1 = -0.7$, and (c) $p_1 = -0.8, q_1 = 0.3$. (d)–(f) are 2D outlines for (a)–(c) concerning contours in ξy-plane.

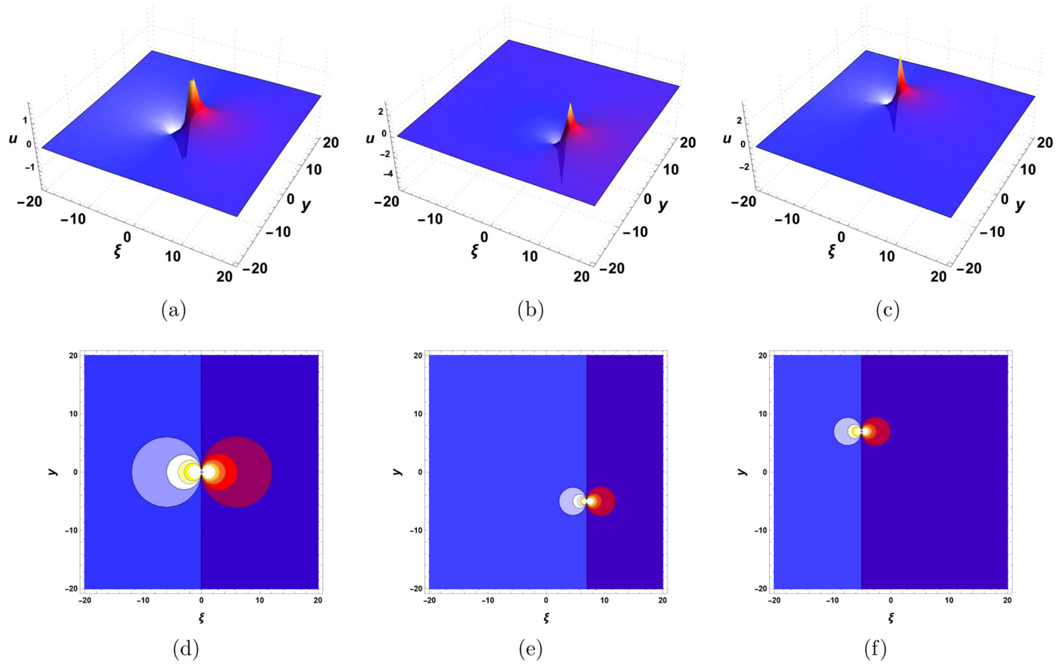

FIG. 2. Rogue waves for first-order solution (29) with (28) having values (a) $u_0 = 0$, $h = 0.1$, $\beta = \gamma = 0$, (b) $u_0 = 0$, $h = 0.1$, $\beta = 7$, $\gamma = -6$, and (c) $u_0 = 0$, $h = 0.1$, $\beta = -5$, $\gamma = 7$. (d)–(f) are 2D outlines for (a)–(c) concerning contours in ξy-plane.

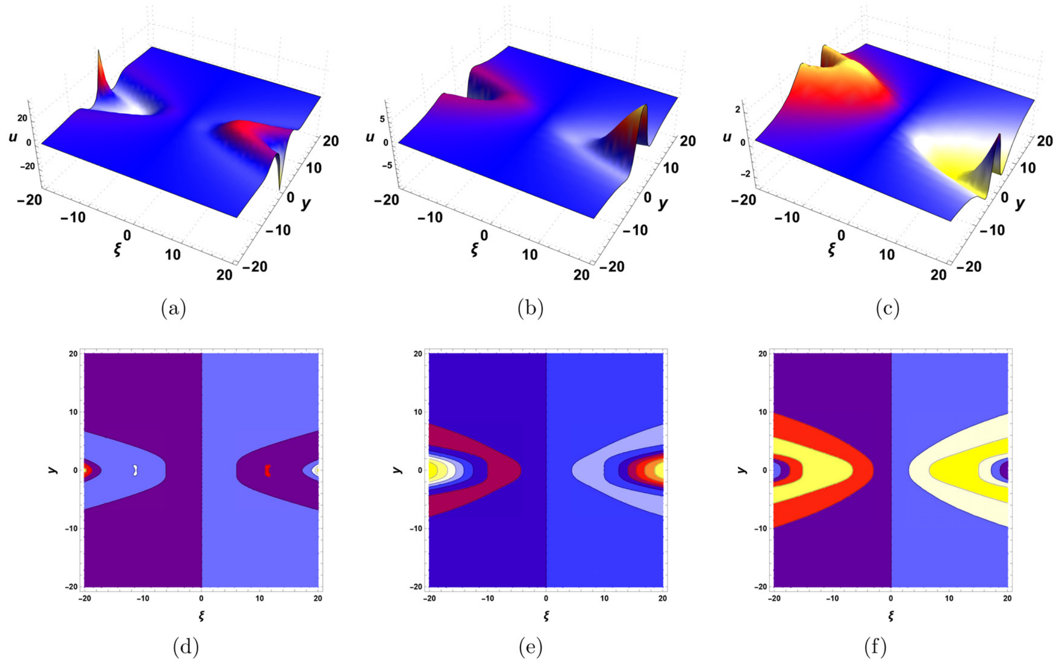

FIG. 3. Rogue waves for second-order solution (33) with (32) having values (a) $u_0 = 0$, $h = 4$, $\beta = \gamma = 0$, (b) $u_0 = 0$, $h = 8$, $\beta = \gamma = 0$, and (c) $u_0 = 0$, $h = 12$, $\beta = \gamma = 0$. (d)–(f) are 2D outlines for (a)–(c) concerning contours in ξy-plane.

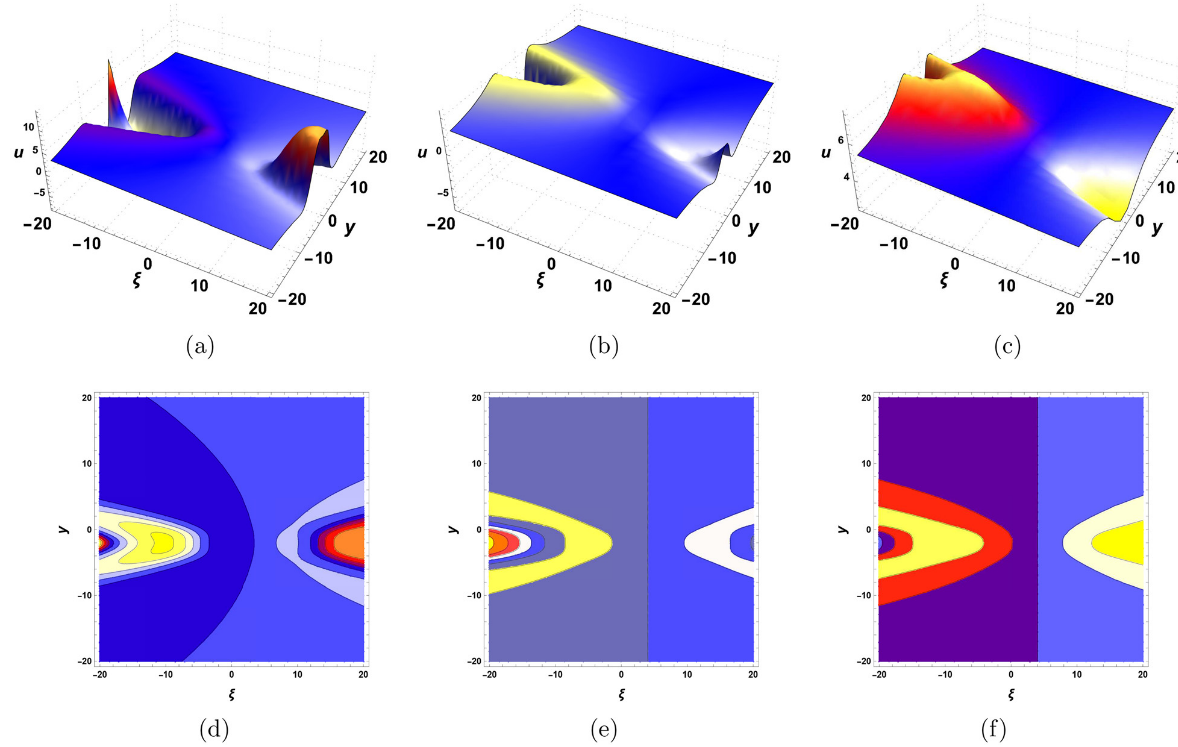

FIG. 4. Rogue waves for second-order solution (33) with (32) having values (a) $u_0 = 2$, $h = 5$, $\beta = 4$, $\gamma = -2$, (b) $u_0 = 2$, $h = 10$, $\beta = 4$, $\gamma = -2$, and (c) $u_0 = 5$, $h = 15$, $\beta = 4$, $\gamma = -2$. (d)–(f) are 2D outlines for (a)–(c) concerning contours in ξy-plane.

$u_0 = 0, h = 0.1, \beta = \gamma = 0$, (b) $u_0 = 0, h = 0.1, \beta = 7, \gamma = -5$, and (c) $u_0 = 0, h = 0.1, \beta = -5, \gamma = 7$.

- In Fig. 3, we show the second-order solution of rogue waves with center parameters (β, γ). Dynamics shows that the direction and amplitude of rogue waves depend on the transforming parameter h in $\xi = x - ht$. Rogue waves are plotted with values (a) $u_0 = 0,\ h = 4,\ \beta = \gamma = 0$, (b) $u_0 = 0,\ h = 8,\ \beta = \gamma = 0$, and (c) $u_0 = 0,\ h = 12,\ \beta = \gamma = 0$.
- Figure 4 shows the second-order solution of rogue waves with center parameters (β, γ). It shows that the direction and amplitude of rogue waves depend on the transforming parameter h in $\xi = x - ht$. Rogue waves are plotted with values (a) $u_0 = 2, h = 5, \beta = 4, \gamma = -2$, (b) $u_0 = 2, h = 10, \beta = 4, \gamma = -2$, and (c) $u_0 = 5, h = 15, \beta = 4, \gamma = -2$.

V. CONCLUSIONS

In conclusion, our investigation of the $(2+1)$-dimensional SWW equation governing ion-acoustic waves in plasma physics has revealed analytical insights and dynamic phenomena. Through a meticulous analysis of Cole–Hopf transformations in dimensions x, y, and t, we have derived the dispersion relation for the phase variable and illustrated soliton solutions that remain unaffected by these transformations. Our investigation extends to rogue waves, delving into first- and second-order occurrences using a generalized N-rogue wave expression derived by the N-soliton in the Hirota technique. Application of symbolic computation, notably the center parameters β and γ, has allowed us to formulate rogue wave solutions, offering a subtle understanding between the parameters and the resulting dynamics. By employing direct computation for various parameter values and reasonable choices of constants, we have manifested solutions of rogue waves up to second-order with their dynamics. Moreover, our exploration incorporates a logarithmic transformation for the dependent variable u, leading to a trilinear equation in $F(\xi, y)$. In practical terms, our findings resonate across diverse scientific disciplines, ranging from classical shallow water theory and fluid dynamics to optical fibers and nonlinear dynamics. The three-dimensional space investigated in the context of ion-acoustic solitons in plasmas holds promise for real-world applications, offering insight that transcends the boundaries of plasma physics.

This research contributes to the theoretical understanding of the $(2+1)$-dimensional SWW equation with practical applications in diverse nonlinear fields. The dynamics, soliton solutions, and rogue wave occurrences uncovered in this study provide a solid foundation for future investigations and underline the rich potential of this area of research.

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AUTHOR DECLARATIONS

Conflict of interest

The authors have no conflicts to disclose.

Author Contributions

Sachin Kumar: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (lead); Software (equal); Supervision (lead); Validation (equal); Writing—original draft (equal). **Brij Mohan:** Conceptualization (equal); Formal analysis (equal); Investigation (lead); Methodology (lead); Software (lead); Supervision (lead); Validation (equal); Writing–original draft (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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Rogue-wave structures for a generalized (3+1)-dimensional nonlinear wave equation in liquid with gas bubbles

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Abstract

This study explores the behavior of higher-order rogue waves within a (3+1)-dimensional generalized nonlinear wave equation in liquid-containing gas bubbles. It creates the investigated equation's Hirota D-operator bilinear form. We employ a generalized formula with real parameters to obtain the rogue waves up to the third order using the direct symbolic technique. The analysis reveals that the second and third-order rogue solutions produce two and three-waves, respectively. To gain deeper insights, we use the Cole-Hopf transformation on the transformed variables ξ and η to produce a bilinear equation. Using the system software *Mathematica*, the dynamic analysis presents the graphics for the obtained solutions in transformed ξ, η, and original spatial-temporal coordinates x, y, z, t. These visualizations reveal rogue waves' intricate structure and evolution, capturing their localized interactions and significant presence within nonlinear systems. We demonstrate that rogue waves, characterized by their substantial height and sudden appearance, are prevalent in various nonlinear events. The equation examined in this study offers valuable insights into the evolution of longer waves with smaller amplitudes, which is particularly relevant in fields such as fluid dynamics, dispersive media, and plasmas. The implications of this research extend across multiple scientific domains, including fiber optics, oceanography, dusty plasma, and nonlinear systems, where understanding the behavior of rogue waves is crucial for both theoretical and practical applications.

1. Introduction

Partial differential equations (PDEs) [1–9] containing dependent variable functions and their partial derivatives are a significant topic in applied mathematics and mathematical physics. Several nonlinear sciences and engineering fields employ PDEs to represent complex physical procedures. Mathematicians have utilized nonlinear PDEs to explain various scientific phenomena, including gravitational research and fluid dynamics. Analyzing and solving nonlinear PDEs can be challenging because no universal method exists. In many different nonlinear sciences, PDEs represent and comprehend physical phenomena that contain numerous variables and their derivatives. The wave equation [10], heat equation [11], and the well-known Schrödinger equation [12] from quantum mechanics are a few examples of PDEs. The Bäcklund transformation [3, 4], Hirota's bilinearization method [5–7], Darboux transformation [8, 9], inverse scattering method [13, 14], bilinear neural network method [15, 16], simplified Hirota's approach [17, 18], Lie symmetry approach [19–21], and other techniques are used to solve nonlinear evolution equations and obtain the analytical and exact solutions.

Rogue waves, sometimes known as extreme waves [22–32], are large-scale localized waves in space and time. They threaten sea-farers, ships and vessels, and other entities. However, it may also help extract useful information about a system and its behavior in non-oceanic cases. For example, it can lead to the formation of extreme wave localization in optics. The evolution of rogue waves is an important topic for many scholars. Unlike typical ocean waves, rogue waves can reach towering heights of 20-30 meters or more, often appearing unexpectedly in relatively calm seas or during storms, making them extremely dangerous for ships and offshore structures. The height of rogue waves distinguishes them from neighboring waves. Rogue waves are investigated in nonlinear wave dynamics because they violate widely accepted linear wave theories. The goal of scientific study on exceptional or large waves is to predict their occurrence and comprehend the fundamental principles that underlie them. When shorter waves concentrate their energy on a narrow area, rogue waves occur from nowhere. The sea safety is one such improvement use. In order to prevent the harm that these waves can impose, the evolving models and algorithms give early warnings and detections. The marine sector, the coastal region, and offshore oil and gas areas could all benefit from knowing this information. Thus, knowing the mechanics of rogue waves helps develop safe structures and mitigation techniques. Thus, getting operational safety and reasonable solutions is feasible. Furthermore, examining the dynamics and origins adds to our understanding of complicated procedures and the formation of extreme processes in various nonlinear sciences.

In this work, we examine newly constructed rogue waves of a generalized (3+1)-dimensional nonlinear wave equation [33, 34] in liquid with gas bubbles as

$$(u_t + \alpha_1 u u_x + \alpha_2 u_{xxx} + \alpha_3 u_x)_x + \alpha_4 u_{yy} + \alpha_5 u_{zz} = 0, \tag{1}$$

where $u(x, y, z, t)$ is a wave function representing the amplitude of the propagating wave as a function of space and time, and $\alpha_{1\leqslant i\leqslant 5}$ are non-zero constants. The nonlinear term $\alpha_1 u u_x$ represents the self-interaction of the wave, which captures the formation of rogue waves, vast and unexpected waves that occur in the nonlinear system. The term $\alpha_2 u_{xxx}$ accounts for the dispersive effects in the medium, where different waves travel at different speeds, conducting the spreading of the wave packet. The dissipative term $\alpha_3 u_x$ represents the energy loss or dissipation in the medium due to factors like viscosity in the liquid and the cross-diffusion terms $\alpha_4 u_{yy}$ and $\alpha_5 u_{zz}$ are a diffusion of the wave in the transverse directions y and z describing the nature of wave propagation.

This research concentrates on originating rogue wave solutions to this generalized nonlinear evolution equation and investigating their dynamics. The rogue waves in this model emphasize the possibility for sudden, large amplitude waves in the liquid medium with gas bubbles. This is relevant in comprehending underwater explosions, sonic booms, or extreme oceanic rogue waves. By exploring the dynamics of these rogue waves, the study provides wisdom into how such waves form, evolve and dissipate over time. This learning can potentially guide the development of processes to predict and mitigate the consequences of rogue waves in real-world scenarios, offering hope in the face of these unforeseen natural phenomena. The relevance of this research to understanding and predicting rogue waves in various scenarios keeps the researchers engaged and interested in the topic. Overall, the physical relevance of this model lies in its ability to capture complex wave phenomena in a nonlinear medium with multiple interacting effects (nonlinearity, dispersion, dissipation, and cross-diffusion), providing a more profound understanding of the mechanisms after rogue wave appearance and propagation.

The equation (1) models the propagation of waves in a liquid medium containing gas bubbles. This equation accounts for the complex behavior of waves as they interact with the bubbles within the liquid, capturing the effects of non-linearity, dispersion, and scattering in a three-dimensional space over time. Such modeling is crucial in various physical and engineering applications, including underwater acoustics, bio-medical ultrasound, and industrial processes involving cavitation. The exact solutions and symmetry reductions explored in the studies [33, 34] offer valuable insights into the fundamental dynamics of these wave phenomena, making the equation a vital tool for predicting and understanding how waves behave in bubbly liquids. The associated conservation laws also ensure that the model adheres to essential physical principles, such as energy conservation and momentum, further validating its applicability in real-world scenarios. This equation generalizes the well-known equations from different areas of nonlinear science as

- (3+1)-dimensional nonlinear wave equation [35] for $\alpha_1 = 1,\ \alpha_2 = \frac{1}{4},\ \alpha_3 = -1$, and $\alpha_4 = \alpha_5 = \frac{3}{4}$ as

$$\left(u_t + u u_x + \frac{1}{4} u_{xxx} - u_x\right)_x + \frac{3}{4}(u_{yy} + u_{zz}) = 0, \tag{2}$$

- (3+1)-dimensional Kadomtsev-Petviashvili equation [36] for $\alpha_1 = -6, \alpha_2 = 1, \alpha_3 = 0$, and $\alpha_4 = \alpha_5 = 3$ as

$$(u_t - 6 u u_x + u_{xxx})_x + 3(u_{yy} + u_{zz}) = 0, \tag{3}$$

- (3+1)-dimensional nonlinear wave equation [37] for $\alpha_1 = \alpha_2 = 1, \alpha_3 = 0$, and $\alpha_4 = \alpha_5 = \frac{1}{2}$ as

$$(u_t + u u_x + u_{xxx})_x + \frac{1}{2}(u_{yy} + u_{zz}) = 0, \tag{4}$$

- (2+1)-dimensional Kadomtsev-Petviashvili equation [38] for $\alpha_1 = \alpha_2 = 1, \alpha_3 = 0, \alpha_4 = 1$, and $\alpha_5 = 0$ as

$$(u_t + uu_x + u_{xxx})_x + u_{yy} = 0, \tag{5}$$

- (1+1)-dimensional Korteweg-de Vries equation [39] for $\alpha_1 = \alpha_2 = 1$, and $\alpha_3 = \alpha_4 = \alpha_5 = 0$ as

$$u_t + uu_x + u_{xxx} = 0, \tag{6}$$

These equations (2)–(6), derived from various physical systems, often describe wave phenomena. The KdV-type equations are crucial in the field of plasma physics, as they provide insights into the behavior and structures of waves. Our study is significant as it constructs the bilinear form for the examined equation in transformed variables and uses a direct symbolic technique to visually evaluate the new rogue waves. This technique transforms the studied equation into a new (1+1)-dimensional evolution equation in the transformed variables. Importantly, we show that the investigated equation can be transformed into a bilinear form in the auxiliary function using the Cole-Hopf transformation, which has practical implications for understanding and predicting nonlinear wave behavior in plasmas.

Our study advances by deriving higher-order rogue wave solutions for investigating (3+1)-dimensional generalized nonlinear wave equation in liquid-containing gas bubbles using its Hirota bilinear form. While previous studies [33–38] have primarily focused on lower-order rogue waves or different nonlinear solutions, our research extends this understanding to more complex scenarios involving second and third-order rogue waves. This is particularly significant from a mathematical viewpoint, as it applies advanced techniques like the Cole-Hopf transformation and direct symbolic methods to obtain and analyze these solutions. From a physical perspective, our model's depiction of higher-order rogue waves is crucial as it reveals the intricate dynamics and interactions that occur in nonlinear systems, which are not captured by lower-order solutions. These higher-order rogue waves provide a more comprehensive understanding of the evolution of large waves from smaller amplitudes, which is essential for accurately modeling real-world phenomena in various scientific domains, including oceanography, dusty plasma, and fiber optics.

In this manuscript, the next section details the direct symbolic technique for identifying the solutions for rogue waves to the analyzed equation. It involves using the Cole-Hopf transformation in transformed variables to obtain a bilinear equation and determines the rogue waves up to the 3rd-order along with their dynamics. Section 3 will present and discuss the results and findings, while the final section will conclude the research study.

2. Rogue waves via direct symbolic approach

We transform the equation (1) with $\xi = x + t$; $\eta = y + z$ in $u(x, y, z, t) = u(\xi, \eta)$ as

$$u_{\xi\xi} + \alpha_1(uu_{\xi\xi} + u_\xi^2) + \alpha_2 u_{\xi\xi\xi\xi} + \alpha_3 u_{\xi\xi} + \alpha_4 u_{\eta\eta} + \alpha_5 u_{\eta\eta} = 0. \tag{7}$$

Taking the phase $\theta_{i\in N}$ in equation (7) as

$$\theta_i = p_i\xi - w_i\eta, \tag{8}$$

having constants $p_{i\in N}$ and dispersions $w_{i\in N}$. In linear terms of equation (7), having $u(\xi,\ \eta) = e^{\theta_i}$ gives

$$w_i = \frac{\pm i p_i\sqrt{1 + \alpha_3 + \alpha_2 p_i^2}}{\sqrt{\alpha_4 + \alpha_5}}. \tag{9}$$

Considering the dependent variable transformation as

$$u(\xi,\ \eta) = K(\log f)_{\xi\xi}, \tag{10}$$

with nonzero constant K and auxiliary function $f(\xi, \eta)$. Substitution of equation (10) with $f = 1 + e^{\theta_1}$ in equation (7) gives

$$K = \frac{12\alpha_2}{\alpha_1}.$$

Thus, the equation (7) can be transformed using the transformation (10) into a bilinear equation in f as

$$\alpha_2(ff_{\xi\xi\xi\xi} - 4f_\xi f_{\xi\xi\xi} + 3f_{\xi\xi}^2) + (1 + \alpha_3)(ff_{\xi\xi} - f_\xi^2) + (\alpha_4 + \alpha_5)(ff_{\eta\eta} - f_\eta^2) = 0, \tag{11}$$

that further gives a bilinear form in Hirota D-operator [39] as

$$[\alpha_2 D_\xi^4 + (1+\alpha_3)D_\xi^2 + (\alpha_4+\alpha_5)D_\eta^2]f.f = 0, \tag{12}$$

where differential operators $D_{i=x,y}$ is defined as

$$D_x^{r_1}D_y^{r_2}U(x, y)V(x, y) = \left(\frac{\partial}{\partial_x} - \frac{\partial}{\partial_{x'}}\right)^{r_1}\left(\frac{\partial}{\partial_y} - \frac{\partial}{\partial_{y'}}\right)^{r_2} U(x, y)V(x', y')|_{x=x', y=y'},$$

with formal variables x', y' and positive integers $r_{i=1,2}$.

We construct the rogue waves by assuming the function f [40, 41] as

$$f(\xi, \eta) = \sum_{q=0}^{\frac{n(n+1)}{2}}\sum_{i=0}^{q} q_{n(n+1)-2q,2i}\,\xi^{n(n+1)-2q}\eta^{2i}, \tag{13}$$

where $s_{l,m\in\{0,2,\cdots,q(q+1)\}}$ are the real parameters.

2.1. First-order rogue waves

Having $n = 1$ in equation (13), we get f as

$$f(\xi, \eta) = f_1 = s_{2,0}\xi^2 + s_{0,2}\eta^2 + s_{0,0}. \tag{14}$$

We get a system of equations by putting the equation (14) in equation (11) with equating all coefficients of distinct powers of ξ and η to zero as

$$\begin{aligned} 2(\alpha_4 s_{0,0}s_{0,2} + \alpha_5 s_{0,0}s_{0,2} + s_{2,0}((\alpha_3+1)s_{0,0} + 6\alpha_2 s_{2,0})) = 0,\\ 2s_{2,0}(\alpha_4 s_{0,2} + \alpha_5 s_{0,2} - (\alpha_3+1)s_{2,0}) = 0,\\ 2s_{0,2}(-\alpha_4 s_{0,2} - \alpha_5 s_{0,2} + (\alpha_3+1)s_{2,0}) = 0. \end{aligned} \tag{15}$$

On solving the system gives parameter values as

$$s_{0,0} = -\frac{3\alpha_2 s_{2,0}}{\alpha_3+1}, \qquad s_{0,2} = \frac{\alpha_3 s_{2,0} + s_{2,0}}{\alpha_4+\alpha_5}, \qquad s_{2,0} = s_{2,0}. \tag{16}$$

So, the function (14) becomes

$$f = f_1 = s_{2,0}\left(\frac{(\alpha_3+1)\eta^2}{\alpha_4+\alpha_5} - \frac{3\alpha_2}{\alpha_3+1} + \xi^2\right), \tag{17}$$

So, we get the solution by putting the equation (17) into (10) as

$$u(\xi, \eta) = u_1 = \frac{24\alpha_2\left(\frac{(\alpha_3+1)\eta^2}{\alpha_4+\alpha_5} - \frac{3\alpha_2}{\alpha_3+1} - \xi^2\right)}{\alpha_1\left(\frac{(\alpha_3+1)\eta^2}{\alpha_4+\alpha_5} - \frac{3\alpha_2}{\alpha_3+1} + \xi^2\right)^2}, \tag{18}$$

2.2. Second-order rogue waves

Considering the function f for $n = 2$ in equation (13) as

$$f(\xi, \eta) = f_2 = s_{6,0}\xi^6 + s_{4,2}\eta^2\xi^4 + s_{4,0}\xi^4 + s_{2,4}\eta^4\xi^2 + s_{2,2}\eta^2\xi^2 + s_{2,0}\xi^2 + s_{0,6}\eta^6 + s_{0,4}\eta^4 + s_{0,2}\eta^2 + s_{0,0}. \tag{19}$$

Substitution of equation (19) into (12) gives a system on equating to zero the coefficients of distinct powers of ξ and η. On solving, we get the parameters as

$$\begin{aligned} s_{0,0} = -\frac{625\alpha_2^3(\alpha_4+\alpha_5)s_{4,2}}{(\alpha_3+1)^4}, \quad s_{0,2} = \frac{475\alpha_2^2 s_{4,2}}{3(\alpha_3+1)^2}, \quad s_{0,4} = -\frac{17\alpha_2 s_{4,2}}{3(\alpha_4+\alpha_5)}, \quad s_{0,6} = \frac{(\alpha_3+1)^2 s_{4,2}}{3(\alpha_4+\alpha_5)^2},\\ s_{2,0} = -\frac{125\alpha_2^2(\alpha_4+\alpha_5)s_{4,2}}{3(\alpha_3+1)^3}, \quad s_{2,2} = -\frac{30\alpha_2 s_{4,2}}{\alpha_3+1}, \quad s_{2,4} = \frac{(\alpha_3+1)s_{4,2}}{\alpha_4+\alpha_5}, \quad s_{4,0} = -\frac{25\alpha_2(\alpha_4+\alpha_5)s_{4,2}}{3(\alpha_3+1)^2},\\ s_{6,0} = \frac{(\alpha_4+\alpha_5)s_{4,2}}{3(\alpha_3+1)}, \end{aligned} \tag{20}$$

Therefore, the function (19) becomes

$$\begin{aligned} f(\xi, \eta) = f_2 = \frac{s_{4,2}}{3}\Bigg(\frac{25\alpha_2^2(19\alpha_3\eta^2 - 5\alpha_4\xi^2 - 5\alpha_5\xi^2 + 19\eta^2)}{(\alpha_3+1)^3} + \frac{(\alpha_3\eta^2 + \alpha_4\xi^2 + \alpha_5\xi^2 + \eta^2)^3}{(\alpha_3+1)(\alpha_4+\alpha_5)^2}\\ - \alpha_2\left(\frac{17\eta^4}{\alpha_4+\alpha_5} + \frac{90\eta^2\xi^2}{\alpha_3+1} + \frac{25\alpha_4\xi^4}{(\alpha_3+1)^2} + \frac{25\alpha_5\xi^4}{(\alpha_3+1)^2}\right) - \frac{1875(\alpha_4+\alpha_5)\alpha_2^3}{(\alpha_3+1)^4}\Bigg), \end{aligned} \tag{21}$$

which gives the solution on substituting it into (10)

$$u(\xi, \eta) = u_2 = \frac{12\alpha_2}{\alpha_1}(\log f_2)_{\xi\xi}. \tag{22}$$

2.3. Third-order rogue waves

Having $n = 3$ in equation (13) gives the function f as

$$\begin{aligned} f(\xi, \eta) = f_3 = {} & s_{0,0} + s_{0,2}\eta^2 + s_{0,4}\eta^4 + s_{0,6}\eta^6 + s_{0,8}\eta^8 + s_{0,10}\eta^{10} + s_{0,12}\eta^{12} + s_{2,0}\xi^2 + s_{2,2}\xi^2\eta^2 + s_{2,4}\xi^2\eta^4 \\ & + s_{2,6}\xi^2\eta^6 + s_{2,8}\xi^2\eta^8 + s_{2,10}\xi^2\eta^{10} + s_{4,0}\xi^4 + q_{4,2}\xi^4\eta^2 + s_{4,4}\xi^4\eta^4 + s_{4,6}\xi^4\eta^6 + s_{4,8}\xi^4\eta^8 + s_{6,0}\xi^6 \\ & + s_{6,2}\xi^6\eta^2 + s_{6,4}\xi^6\eta^4 + s_{6,6}\xi^6\eta^6 + s_{8,0}\xi^8 + s_{8,2}\xi^8\eta^2 + s_{8,4}\xi^8\eta^4 + s_{10,0}\xi^{10} + s_{10,2}\xi^{10}\eta^2 + s_{12,0}\xi^{12}. \end{aligned} \tag{23}$$

On putting the equation (23) into (12), we get a system on equating to zero the coefficients of distinct powers of ξ and η. On solving the system, we get the parameters as

$$\begin{aligned} & s_{0,0} = \frac{878826025\alpha_2^6(\alpha_4 + \alpha_5)s_{10,2}}{54(\alpha_3 + 1)^7}, \quad s_{0,2} = -\frac{150448375\alpha_2^5 s_{10,2}}{9(\alpha_3 + 1)^5}, \quad s_{0,4} = \frac{16391725\alpha_2^4 s_{10,2}}{18(\alpha_3 + 1)^3(\alpha_4 + \alpha_5)}, \\ & s_{0,6} = -\frac{399490\alpha_2^3 s_{10,2}}{9(\alpha_3 + 1)(\alpha_4 + \alpha_5)^2}, \quad s_{0,8} = \frac{1445\alpha_2^2(\alpha_3 + 1)s_{10,2}}{2(\alpha_4 + \alpha_5)^3}, \quad s_{0,10} = -\frac{29\alpha_2(\alpha_3 + 1)^3 s_{10,2}}{3(\alpha_4 + \alpha_5)^4}, \\ & s_{0,12} = \frac{(\alpha_3 + 1)^5 s_{10,2}}{6(\alpha_4 + \alpha_5)^5}, \quad s_{2,0} = -\frac{79893275\alpha_2^5(\alpha_4 + \alpha_5)s_{10,2}}{9(\alpha_3 + 1)^6}, \quad s_{2,2} = \frac{94325\alpha_2^4 s_{10,2}}{(\alpha_3 + 1)^4}, \\ & s_{2,4} = \frac{2450\alpha_2^3 s_{10,2}}{(\alpha_3 + 1)^2(\alpha_4 + \alpha_5)}, \quad s_{2,6} = \frac{17710\alpha_2^2 s_{10,2}}{3(\alpha_4 + \alpha_5)^2}, \quad s_{2,8} = -\frac{95\alpha_2(\alpha_3 + 1)^2 s_{10,2}}{(\alpha_4 + \alpha_5)^3}, \\ & s_{2,10} = \frac{(\alpha_3 + 1)^4 s_{10,2}}{(\alpha_4 + \alpha_5)^4}, \quad s_{4,0} = -\frac{5187875\alpha_2^4(\alpha_4 + \alpha_5)s_{10,2}}{18(\alpha_3 + 1)^5}, \quad s_{4,2} = -\frac{36750\alpha_2^3 s_{10,2}}{(\alpha_3 + 1)^3}, \\ & s_{4,4} = \frac{18725\alpha_2^2 s_{10,2}}{3(\alpha_3 + 1)(\alpha_4 + \alpha_5)}, \quad s_{4,6} = -\frac{730\alpha_2(\alpha_3 + 1)s_{10,2}}{3(\alpha_4 + \alpha_5)^2}, \quad s_{4,8} = \frac{5(\alpha_3 + 1)^3 s_{10,2}}{2(\alpha_4 + \alpha_5)^3}, \\ & s_{6,0} = -\frac{37730\alpha_2^3(\alpha_4 + \alpha_5)s_{10,2}}{9(\alpha_3 + 1)^4}, \quad s_{6,2} = \frac{9310\alpha_2^2 s_{10,2}}{3(\alpha_3 + 1)^2}, \quad s_{6,4} = -\frac{770\alpha_2 s_{10,2}}{3(\alpha_4 + \alpha_5)}, \\ & s_{6,6} = \frac{10(\alpha_3 + 1)^2 s_{10,2}}{3(\alpha_4 + \alpha_5)^2}, \quad s_{8,0} = \frac{245\alpha_2^2(\alpha_4 + \alpha_5)s_{10,2}}{2(\alpha_3 + 1)^3}, \quad s_{8,2} = -\frac{115\alpha_2 s_{10,2}}{\alpha_3 + 1}, \\ & s_{8,4} = \frac{5(\alpha_3 + 1)s_{10,2}}{2(\alpha_4 + \alpha_5)}, \quad s_{10,0} = -\frac{49\alpha_2(\alpha_4 + \alpha_5)s_{10,2}}{3(\alpha_3 + 1)^2}, \quad s_{10,2} = s_{10,2}, \\ & s_{12,0} = \frac{(\alpha_4 + \alpha_5)s_{10,2}}{6(\alpha_3 + 1)}. \end{aligned} \tag{24}$$

We obtain the solution by putting the equation (23) with the values (24) into (10) as

$$u(\xi, \eta) = u_3 = \frac{12\alpha_2}{\alpha_1}(\log f_3)_{\xi\xi}. \tag{25}$$

3. Results and analysis

This work studied the rogue waves as an extraordinary oceanic phenomenon characterized by their immense height, steepness, and sudden appearance. Their formation shows the link to nonlinear interactions, where energy from smaller waves combines through constructive interference or the interaction between ocean currents and opposing waves. Despite their rarity, rogue waves concentrate immense power in a small area capable of causing catastrophic damage. Predicting these waves remains a significant and ongoing challenge in oceanography, though advances in wave modeling and satellite technology continue to improve our understanding of these formidable forces of nature. The investigated equation showed the rogue wave structures in transformed variables ξ and η with appropriate parameter values utilizing direct symbolic approach. The first-order rogue solution generated a single rogue wave solution, and second and third-order rogue solutions gave the interactions of two and three rogue waves, respectively. The dynamics of rogue wave solutions have been

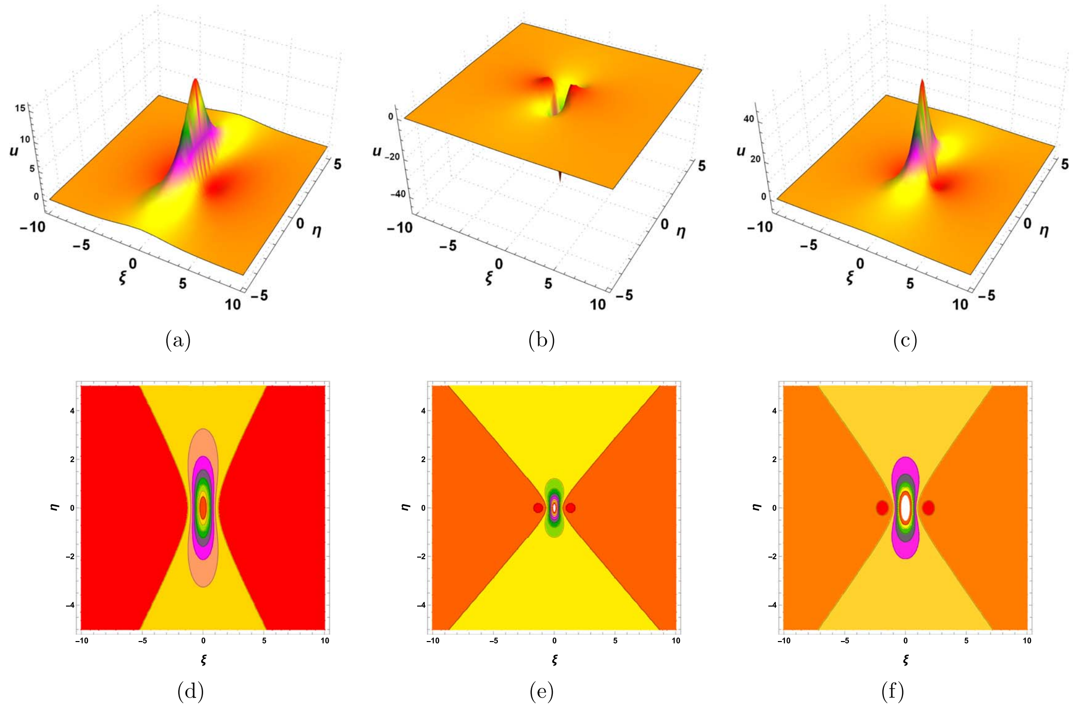

Figure 1. Dynamics of 1st-order rogue waves for (18) in transformed variables ξ and η. (d)–(f) are the contours for (a)–(c) in $\xi\eta$-plane.

shown in transformed variables ξ, η, and in the starting variables x, y, z, t in $\xi\eta$, xt, and xy planes. The analytical and dynamical findings are as follows:

- Figure 1 and 2 depicts the single rogue waves of first order having singularities at $\xi = \eta = 0$. For all three plots in figure 1, the positive and negative direction of ξ shows the bright and the dark parts of the rogue wave dynamics. Depending on the constant parameters, the rogue wave shows the nature of steeped and immense height from 15 to 40 units of u in the forming local region. Figure 2 illustrates the single rogue wave structures in the original variables x, y, z, t. (a)–(c) shows the soliton nature w.r.t. time variable in xt-plane with spacial coordinates as $y = z = 0$, and (d)–(f) shows the single rogue waves in xy-plane with $z = t = 3$. The showed rogue waves for both figures have the parameter values as (**a**) $\alpha_1 = -1, \alpha_2 = -1, \alpha_3 = \alpha_4 = \alpha_5 = 1$; (**b**) $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 5, \alpha_4 = 1, \alpha_5 = 1$; and (**c**) $\alpha_1 = -1, \alpha_2 = -2, \alpha_3 = 5, \alpha_4 = 1, \alpha_5 = 2$.
- In figures 3 and 4, we illustrate the rogue waves of second order which show the interactions of two rogue waves. For all three graphs in figure 3, the two rogue waves intersect at $\xi = \eta = 0$ with having a void area between their bight and dark wave parts which makes them dangerous to sail the ships near them. The interaction of these two rogue waves is dominating each other to form a larger wave than the small waves in a relatively small area. Figure 4 shows the two rogue wave structures in the original variables x, y, z, t. (a)–(c) shows the solitoninc nature w.r.t. time variable, in xt-plane with $y = z = 0$, and (d)–(f) shows the two rogue waves in xy-plane with $z = t = 3$. The second-order rogue waves for both figures have the parameter values as (**a**) $\alpha_1 = -1, \alpha_2 = -1, \alpha_3 = \alpha_4 = \alpha_5 = 1$; (**b**) $\alpha_1 = 2, \alpha_2 = -1, \alpha_3 = 2, \alpha_4 = 1, \alpha_5 = 2$; and (**c**) $\alpha_1 = -3, \alpha_2 = -1, \alpha_4 = 1, \alpha_3 = \alpha_5 = 2$.
- Figures 5 and 6 show the 3rd-order rogue waves that depict the three rogue waves having their interactions and creating a void area among their interactions with sharp and steeped waveforms. The interaction of these three rogue waves is dominating each other to form larger waves than the small waves in a relatively small area which makes them harmful than the ordinary waves. For all plots in figure 5, the three rogue waves depict their bright and dark parts on intersections. Figure 6 shows the three rogue wave structures in the original variables x, y, z, and t. (a)-(c) shows the soliton behavior with respect to the time variable in the xt-plane with $y = z = 0$, and (d)-(f) shows the three rogue waves in the xy-plane. with xy-plane with $z = t = 3$. The showed third-order rogue waves for both figures have the parameter values as (**a**) $\alpha_1 = -3, \alpha_2 = -2, \alpha_3 = \alpha_4 = 2, \alpha_5 = 1$; (**b**) $\alpha_1 = 3, \alpha_2 = -2, \alpha_3 = \alpha_4 = 1, \alpha_5 = 3$; and (**c**) $\alpha_1 = -3, \alpha_2 = -2, \alpha_3 = 4, \alpha_4 = 1, \alpha_5 = 3$.

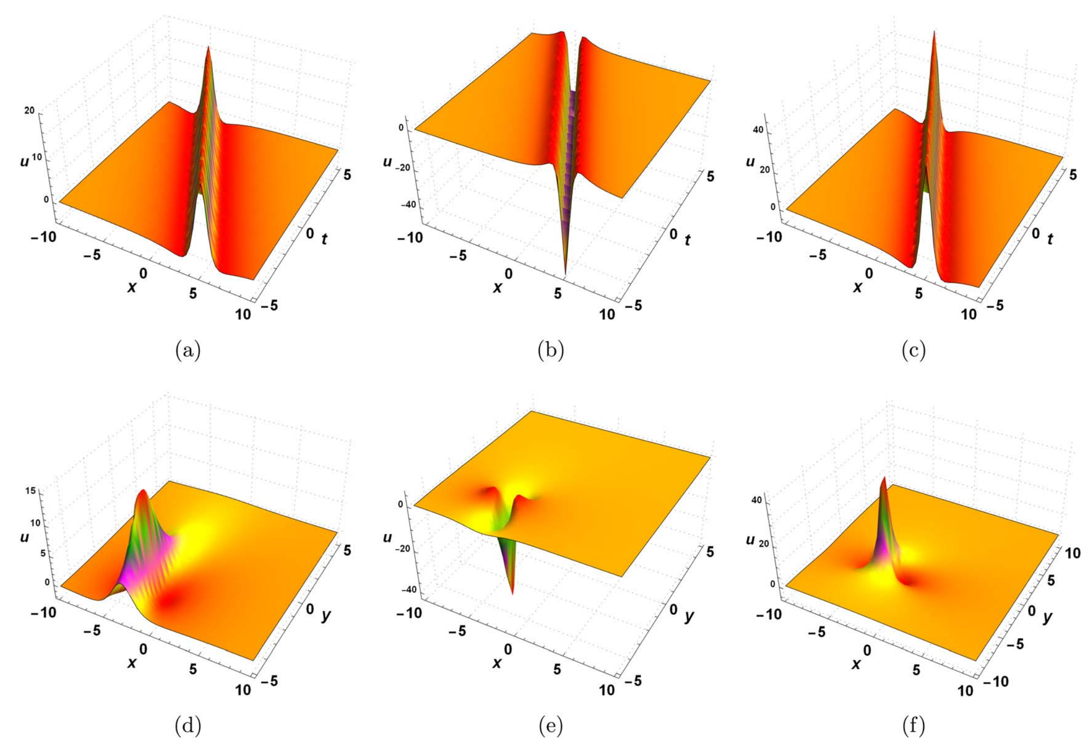

Figure 2. Dynamical profiles of (18) in the starting variables x, y, z, t under transformations $\xi = x + t$ and $\eta = y + z$. (a)–(c) and (d)–(f) depict 3D profiles in xt- and xy-planes, respectively.

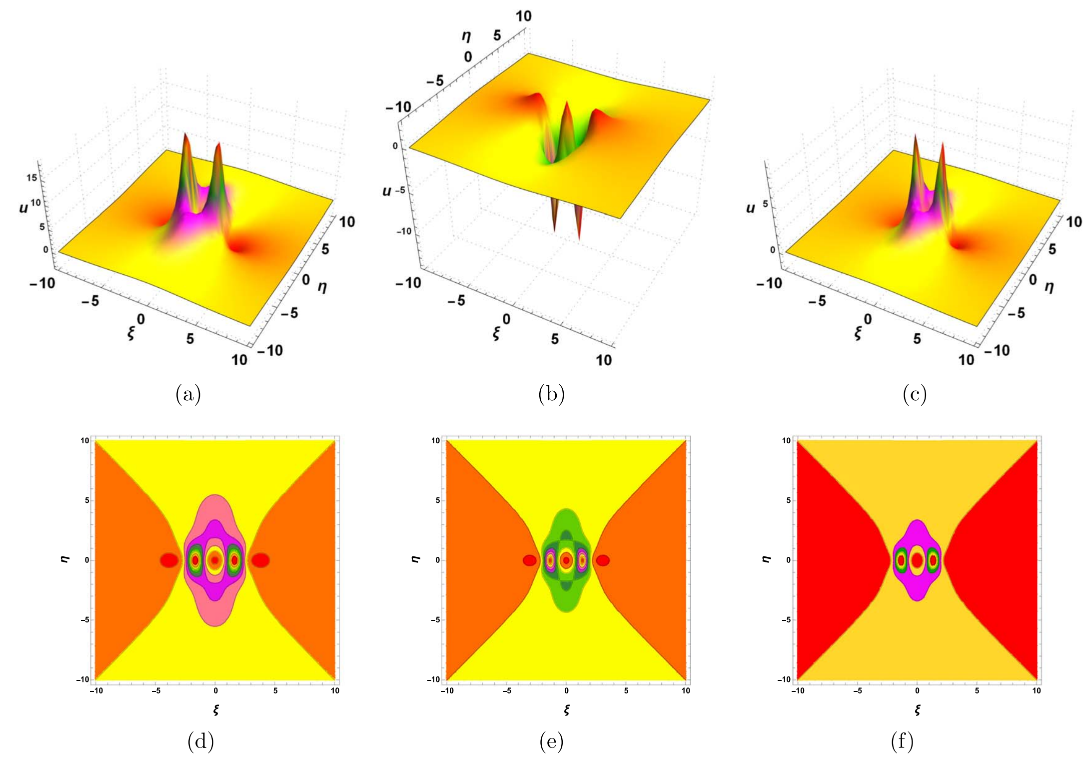

Figure 3. Dynamics of 2nd-order rogue waves for (22) in transformed variables ξ and η. (d)–(f) are the contours for (a)–(c) in $\xi\eta$-plane.

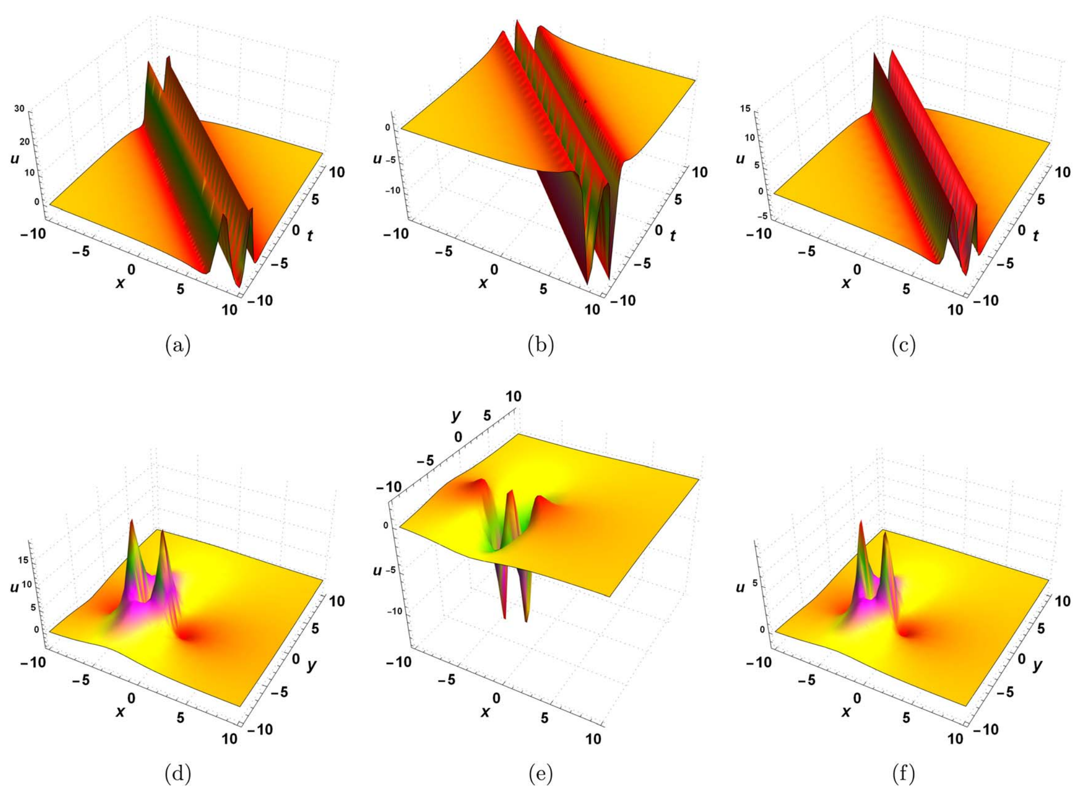

Figure 4. Dynamical profiles of (22) in the starting variables x, y, z, t under transformations $\xi = x + t$ and $\eta = y + z$. (a)–(c) and (d)–(f) depict 3D profiles in xt- and xy-planes, respectively.

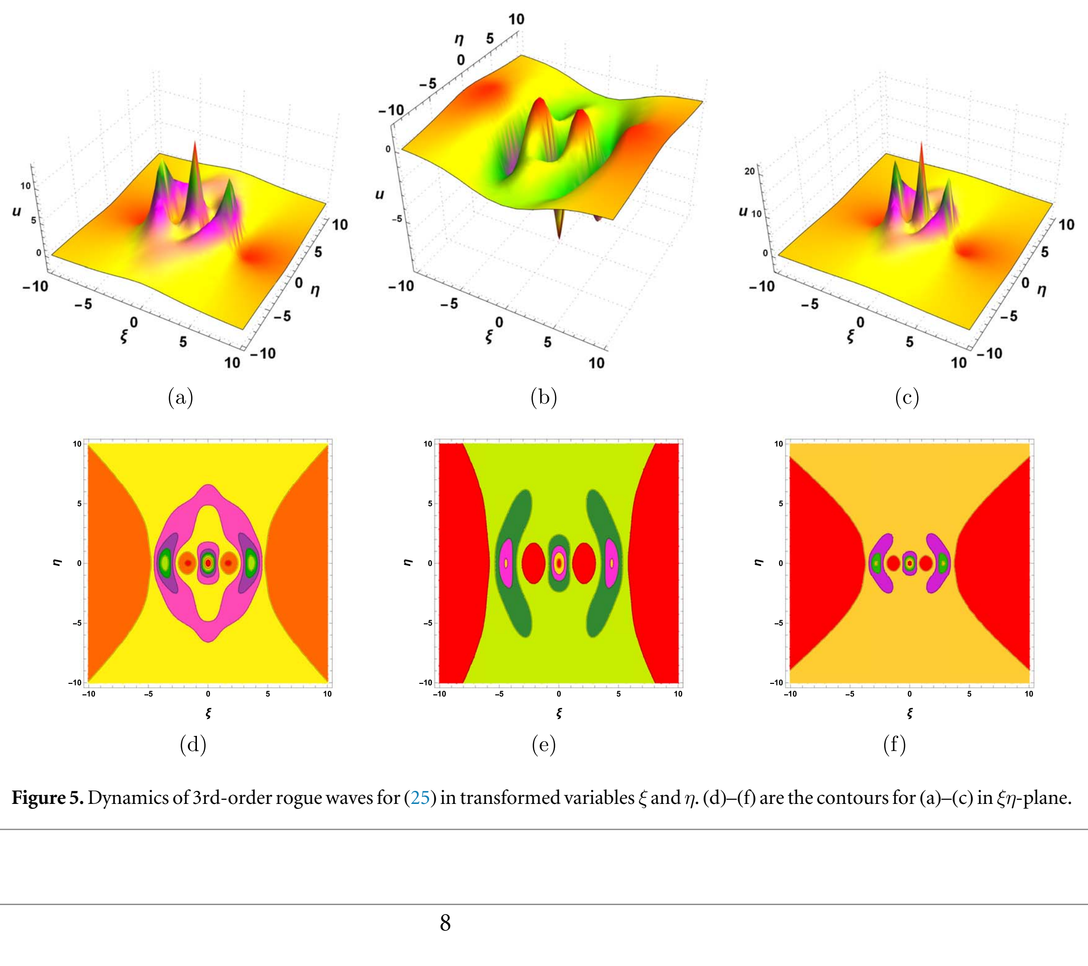

Figure 5. Dynamics of 3rd-order rogue waves for (25) in transformed variables ξ and η. (d)–(f) are the contours for (a)–(c) in $\xi\eta$-plane.

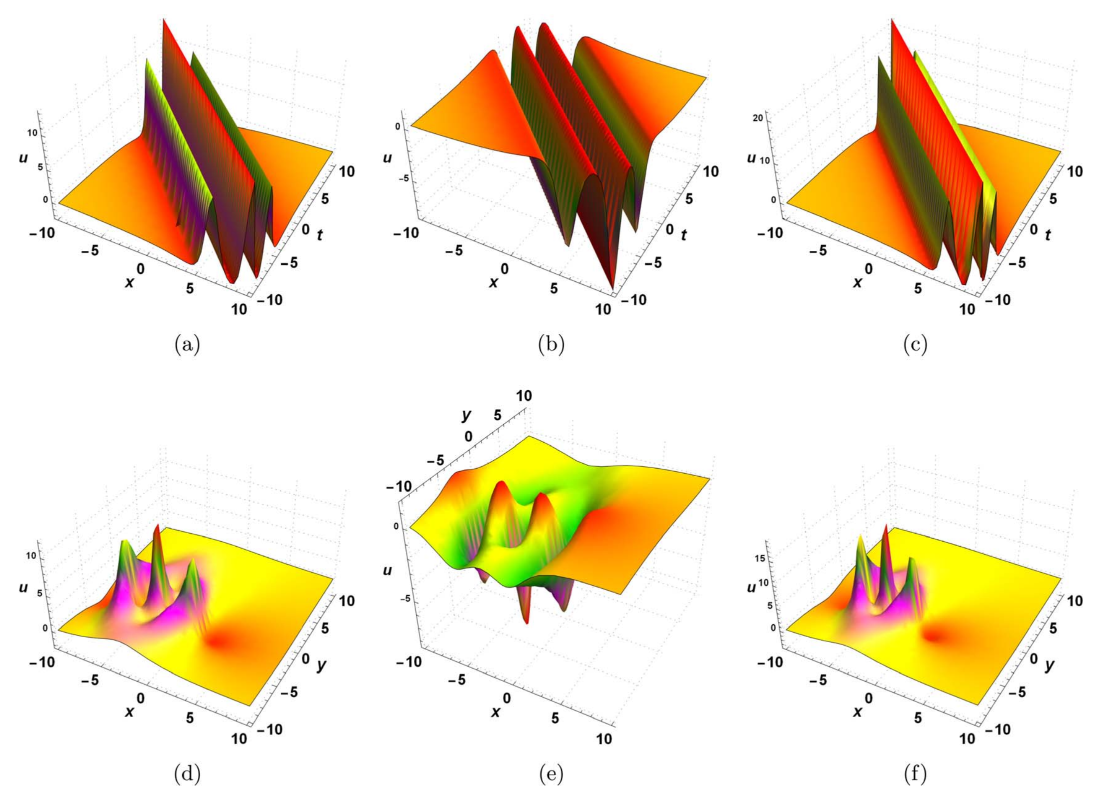

Figure 6. Dynamical profiles of (25) in the starting variables x, y, z, t under transformations $\xi = x + t$ and $\eta = y + z$. (a)–(c) and (d)–(f) depict 3D profiles in xt- and xy-planes, respectively.

4. Conclusions

This study successfully derived higher-order rogue wave solutions for a (3+1)-dimensional generalized nonlinear wave equation in liquid-containing gas bubbles using its Hirota bilinear form. We obtained rogue waves up to the third order through the direct symbolic technique, revealing that second and third-order solutions generate two and three rogue waves, respectively. By applying the Cole-Hopf transformation, we transformed variables ξ and η to produce a bilinear equation, facilitating dynamic analysis using *Mathematica*. The graphical representations in the transformed and original variables illustrate the complex dynamics and interactions of rogue waves in nonlinear systems. Our findings highlight the significant presence and intricate behavior of rogue waves, underlining their importance in understanding the evolution of large waves from smaller amplitudes. These insights are particularly relevant in nonlinear dynamics, dispersive media, and plasma physics.

The implications of this research extend across multiple scientific domains, including dusty plasma, oceanography, fiber optics, and other complex nonlinear systems. By deepening our understanding of rogue wave phenomena, this study contributes to the broader knowledge base. It paves the way for future explorations and applications in fields where critical nonlinear events are pivotal.

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Data availability statement

No new data were created or analyzed in this study.

Declarations

The authors declare the following:

Ethics approval and consent to participate

Not applicable.

Competing interests

The authors state that there is no conflict of interest.

Author's contributions

Both authors have agreed and given their consent for the publication of this research paper.

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