Lump, soliton, and interaction solutions to a generalized two-mode higher-order nonlinear evolution equation in plasma physics

Sachin Kumar[a,*], **Brij Mohan**[b,*], **Raj Kumar**[c]

[a]*Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi-110007, India*
[b]*Department of Mathematics, Hansraj College, University of Delhi, Delhi-110007, India*
[c]*Department of Mathematics, Kirori Mal College, University of Delhi, Delhi-110007, India*

Abstract

This article investigates a nonlinear fifth-order partial differential equation (PDE) in two-mode waves. The equation generalizes two-mode Sawada-Kotera (tmSK), two-mode Lax (tmLax), and two-mode Caudrey-Dodd-Gibbon (tmCDG) equations. In 2017, Wazwaz [1] presented three two-mode fifth-order evolutions equations as tmSK, tmLax, and tmCDG equations for the integrable two-mode KdV equation and established solitons up to three-soliton solutions. In light of the research above, we examine a generalized two-mode evolution equation using a logarithmic transformation concerning the equation's dispersion. It utilizes the simplified technique of the Hirota method to obtain the multiple solitons as a single soliton, two solitons, and three solitons with their interactions. Also, we construct one-lump solutions and their interaction with a soliton and depict the dynamical structures of the obtained solutions for solitons, lump, and their interactions. We show the 3D graphics with their contour plots for the obtained solutions by taking suitable values of the parameters presented in the solutions. These equations simultaneously study the propagation of two-mode waves in the identical direction with different phase velocities, dispersion parameters, and nonlinearity. These equations have applications in several real-life examples, such as gravity-affected waves or gravity-capillary waves, waves in shallow water, propagating waves in fast-mode and the slow-mode with their phase velocity in a strong and weak magnetic field, known as magneto-sound propagation in plasmas.

Keywords: Generalized two-mode evolution equation; Simplified Hirota technique; Multiple solitons; Lump solution; Dependent variable transformation.

1. Introduction

1.1. Motivation

In recent years, the study of the propagation of two-mode different waves in the identical direction simultaneously [1–6] with different phase velocities, nonlinearity, and dispersion parameters have attracted many researchers and investigators to understand the physical significance of such a model in gravity-capillary waves, shallow water waves, and magneto-sound propagation in plasma physics. This model was first noted in 1994 by the Korsunsky [2] to show the deriving of the KdV equation in two-mode, which is a PDE of second-order in temporal coordinate and governs the propagation of two-mode

*Corresponding authors: brijmohan6414@gmail.com (Brij Mohan) and sachinambariya@gmail.com (S. Kumar)

waves in the same direction simultaneously. He proposed the structure of a two-mode equation in a scaled form with a scaled parameter s for any integrable equation as

$$u_{tt} = s^2 u_{xx} - \lambda N(u, u_x, u_{xx}, ...) - \mu L(u_{px}, p \geq 2), \tag{1}$$

where

$$\lambda = (\partial_t - \xi s \partial_x),$$
$$\mu = (\partial_t - \zeta s \partial_x),$$

with

$$|\xi| \leq 1, |\zeta| \leq 1, \partial_k = \frac{\partial}{\partial_k}; k = x, t,$$

are multiplicative factors with respect to spatial and temporal coordinates with nonlinear parameter ξ and the dispersion parameter ζ, and $N = N(u, u_x, u_{xx}, ...)$ and $L = L(u_{px}, p \geq 2)$ are nonlinear and linear terms, respectively. If the second and third terms on the right side of the equation (1) are zero, then it represents the standard wave equation that has a solution as a sum of two waves propagating in the opposite directions. One wave with phase $x - st$ propagates in a positive direction, and another wave with phase $x + st$ propagates in a negative direction. Here, the two-mode wave model propagates two waves in identical directions simultaneously, making it the required field of study. It supports several physical models such as gravity-affected waves or gravity-capillary waves, waves in shallow water, and propagating waves in fast-mode and the slow-mode along with their phase velocity in a strong and weak magnetic field, known as magneto-sound propagation in plasma physics.

1.2. Aim of the research

By using the equation (1), one can derive the two-mode evolution equation for a known nonlinear equation. In 2017, Wazwaz [1] formulated two-mode equations of fifth-order for Sawada-Kotera (SK), Caudrey-Dodd-Gibbon (CDG), and Lax equations and discussed some soliton solutions using the simplified Hirota method. Later in the same year, he proposed a two-mode evolution equation for modified KdV equation [3] and constructed the solutions for multiple solitons using the tanh-coth method. The equations

$$\begin{aligned}
u_{tt} &= s^2 u_{xx} - \lambda \left[\left(\frac{5}{3} u^3 + 5 u u_{xx} \right)_x \right] - \mu u_{5x}, &(2)\\
u_{tt} &= s^2 u_{xx} - \lambda \left[\left(10 u^3 + 10 u u_{xx} + 5 u_x^2 \right)_x \right] - \mu u_{5x}, &(3)\\
u_{tt} &= s^2 u_{xx} - \lambda \left[\left(60 u^3 + 30 u u_{xx} \right)_x \right] - \mu u_{5x}, &(4)
\end{aligned}$$

are the tmSK, tmLax, and tmCDG equations, respectively, where λ and μ stand as in the equation (1). The equations (2), (3), and (4) contain three nonlinear terms each. The terms u_{xx} and $u_{5x} = u_{xxxxx}$ are the disperssive terms in all the above said two-mode equations.

This research investigates a generalized two-mode nonlinear equation of fifth-order for SK, CDG, and Lax equations structured as

$$u_{tt} = s^2 u_{xx} - \lambda \left[\left(C_1 u^3 + C_2 u u_{xx} + C_3 u_x^2 \right)_x \right] - \mu u_{5x}, \tag{5}$$

where C_1, C_2, and C_3 are the constants and $u = u(x, t)$ is the wave surface in x-direction with respect to the time t. The equation (5) generalizes

(i) tmSK equation (2) for $C_1 = \frac{5}{3}, C_2 = 5$, and $C_3 = 0$,
(ii) tmLax equation (3) for $C_1 = 10, C_2 = 10$, and $C_3 = 5$,
(iii) tmCDG equation (4) for $C_1 = 60, C_2 = 30$, and $C_3 = 0$.
We aim to construct the multi-soliton solutions as one, two, and three soliton solutions and their interactions using the simplified technique of the Hirota method and a one-lump solution and its interaction with a single soliton. We also desire to show the dynamical structures of the obtained solutions in 2D and 3D graphics by taking suitable values for the parameters in the solutions for solitons, lump, and their interactions.

1.3. Literature review

Recently, several researchers have established several two-mode equations for different known equations using different techniques and used the abbreviation 'tm' for two-mode before the name of the known equation. In literature review, we found many investigators who examine different two-mode equations for the known equations such as Alquran, Jaradat and Syam [4] studied a two-mode Korteweg–de Vries–Burgers (tmKdVB) equation and obtained soliton solutions using simplified bilinear method and Tanh-coth expension method; Wazwaz [5, 6] observed the kink solutions for two-mode STO (tmSTO) and two-mode fourth-order Burgers (tmfB) equation using simplified Hirota method and also formulated and analysed the multiple soliton solutions for two-wave mode Kadomtsev–Petviashvili (tmKP) equation; Kopçasız, Seadawy and Emrullah [7] found dispersive soliton solutions of two-mode Schrödinger wave equation, using Extended (G'/G) expansion method, Sine-cosine method, and Semi-inverse variational method; Yu, Zhang et al. [8] studied double-lump solution for two-mode optical fiber using Hirota method; Gómez [9] constructed exact solutions for a generalized two-mode KdV (tmKdV) Equation using Improved tanh-coth method; Alquran,Jaradat, et al. [10] studied two-mode KdV–Burgers–Kuramoto (tmKBK) and two-mode Hirota-Satsuma (tmHS) equations to construct the soliton solutions using Tanh-coth expension method and Kudryashov method; Kumar, Park et al. [11] found soliton solutions for two-mode Sawada-Kotera (tmSK) equation using modified Kudryashov method and new auxiliary equation method; Xiao, Tian et al. [12] created multi-soliton solutions for two-mode KdV (tmKdV) equation using Bäcklund transformation; Alquran and Jaradat [13] generated solitary wave solutions for two-mode nonlinear Schrödinger (tmNLS) equation with nonlinearity Kerr laws; Jaradat, Syam, et al. [14] studied a new two-mode coupled Burgers (tmCB)equation and obtain kink solutions using simplified Hirota method; Raza, Jhangeer et al. [15] analysied the wave solutions for two-mode Dynamical analysis and phase portraits of two-mode nonlinear Schrödinger (tmNLS) equation using Exp-expansion method; Ali, Alquran et al. [16] constructed solitary wave solutions for two-mode fifth-order Korteweg–de Vries (tmfKdV) equation using Kudryashov-scheme and Sine–cosine function method; Jaradat, Alquran, et al. [17] studied a two-mode Kuramoto–Sivashinsky (tmKS) equation and formulated soliton solutions using simplified Hirota method and Tanh-expansion method.
In reviews of the works mentioned above, we observed that two-mode nonlinear evolution equations have been fascinating to the mathematicians and physicists to observe the propagation of two different waves in the same direction simultaneously with different phase velocity, nonlinearity, and dispersion. Many techniques have been used to construct different solutions, such as multiple soliton solutions, kink solutions, lump solutions, bright-dark solitons, cross-kink waves, rouge waves and others. The different methods are as the (G'/G)-expansion method, the Lie symmetry method [18–23], the tanh expansion method, the Hirota bilinear method [24–28], the Bilinear neural network method [29–35], the Darboux transformation [36, 37], the Kudryshov method, the sech-csch method, the simplified Hirota method, the sine-cosine method, the Exp-expansion method, the tanh-coth method, the Bäcklund transforma-

tion [38–40], the Tanh-expansion method, the sinh-cosh method, the new auxiliary equation method, and many other techniques [41–43].

1.4. Techniques and objective

The simplified Hirota technique [44–47] is a straightforward interpretation of the Hirota method [48–50] given by Hereman et al. [51]. It is a widely proposed methodology that describes and applies approaches relevant to nonlinear integrable evolution equations. This technique delivers promising results for multiple solitons for a broad scope of nonlinear PDEs. This method does not depend on forming the bilinear form of a nonlinear PDE compared to the Hirota bilinear method; instead, it assumes soliton solutions as polynomials in the expression of the exponential functions.
We seek a common dependent variable transformation for the tmSK, tmLax, and tmCDG equations that fulfill the generalized equation's objective to get a logarithmic transformation. This transformation with the simplified Hirota technique assists in constructing the multi-soliton solutions as one, two, and three solitons and the interactions for the generalized equation. We also construct a one-lump solution and its interaction with a single soliton. For the established lump, soliton, and interaction solutions, we showcase the dynamics through 3D graphics with their contour plots by choosing appropriate values for the parameters in the obtained solutions. These solutions in the form of solitons, lumps, and their interactions can be observed as illustrating the natural phenomena of propagations of waves such as waves in shallow water, waves in narrow channels, gravity-affected waves or gravity-capillary waves, and propagating waves in fast-mode and the slow-mode along with their phase velocity in a strong and weak magnetic field, known as magneto-sound propagation in plasma physics, fiber optics, condensed matter, and others nonlinear sciences.

This article's layout is as follows: Section 2 determines a standard logarithmic transformation for the generalized concerned equation. We utilize the simplified technique of the Hirota method to obtain the solutions for multiple solitons as one, two, and three solitons and the interactions with dynamical structures in 3D and their contour plots in Section 3. Section 4 constructs the lump solution and interaction with a soliton and plots the obtained solutions' graphics. Section 5 discusses the results and findings, and the last Section 6 concludes our work.

2. Transformation of dependent variable

Considering the phase variable Θ_i as

$$\Theta_i = \delta_i x - \eta_i t, \tag{6}$$

where δ_i and η_i are the constants and the dispersion, respectively. We create an equation with the linear terms of Eq. (5) as

$$u_{tt} - s^2 u_{xx} + u_{xxxxxt} - \zeta s u_{xxxxxx} = 0, \tag{7}$$

and substitute $u(x,t) = \mathrm{e}^{\Theta_i}$ into it to get

$$-\zeta s \delta_i^6 \mathrm{e}^{\delta_i x - \eta_i t} - s^2 \delta_i^2 \mathrm{e}^{\delta_i x - \eta_i t} - \delta_i^5 \eta_i \mathrm{e}^{\delta_i x - \eta_i t} + w_i^2 \mathrm{e}^{\delta_i x - \eta_i t} = 0. \tag{8}$$

Now solving the equation (8) for η_i, we get the dispersion which is independent of ξ as

$$\eta_i = \frac{1}{2}\left(\delta_i^5 \pm \delta_i \sqrt{4\zeta s \delta_i^4 + 4s^2 + \delta_i^8}\right). \tag{9}$$

Next, assuming the logarithimic transformation as

$$u = \Lambda(ln\tau)_{2x}, \tag{10}$$

or

$$u = \Lambda\left(\frac{\tau\tau_{xx} - \tau_x^2}{\tau^2}\right), \tag{11}$$

where $\tau = \tau(x, t)$ and Λ are the auxiliary function and constant, respectively. By substituting $\tau = 1+e^{\Theta_i}$ with Eq. (9) into Eq. (5), and solving for Λ, with the scaled parameter s as zero and $\xi = \zeta = 1$, we get
(i) $\Lambda = 6$ for tmSK equation (2) with $C_1 = \frac{5}{3}, C_2 = 5$, and $C_3 = 0$,
(ii) $\Lambda = 2$ for tmLax equation (3) with $C_1 = 10, C_2 = 10$, and $C_3 = 5$,
(iii) $\Lambda = 1$ for tmCDG equation (4) with $C_1 = 60, C_2 = 30$, and $C_3 = 0$.
We create a system with coefficient matrix M for the values of the constants $\{C_1, C_2, C_3\}$, the coulmn vector D with the values of the contant Λ, and the unkown column vector P for the unkowns $\{p_1, p_2, p_3\}$ as

$$MP = D \Rightarrow \begin{pmatrix} 10 & 10 & 5 \\ \frac{5}{3} & 5 & 0 \\ 60 & 30 & 0 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}. \tag{12}$$

We determine the values of p_1, p_2 and p_3 by solving the system (12) as

$$p_1 = -\frac{7}{10}, p_2 = \frac{43}{30}, p_3 = -\frac{16}{15}.$$

So, we get a general value for $\Lambda = p_1C_1 + p_2C_2 + p_3C_3$ as

$$\Lambda = -\frac{1}{30}(21C_1 - 43C_2 + 32C_3),$$

and the dependent variable transformation (11) for the Eq. (5) will be as

$$u(x,t) = -\frac{1}{30}(21C_1 - 43C_2 + 32C_3)\left(\frac{\tau\tau_{xx} - \tau_x^2}{\tau^2}\right). \tag{13}$$

3. Soliton solutions with simplified Hirota method

3.1. Solution for a single soliton

We assume the function τ in Eq. (13) as

$$\tau(x,t) = 1 + \mathrm{e}^{\Theta_1} = 1 + \mathrm{e}^{\left(\delta_1 x - \frac{1}{2}t\left(\delta_1^5 + \sqrt{4\zeta\delta_1^6 s + 4\delta_1^2 s^2 + \delta_1^{10}}\right)\right)}. \tag{14}$$

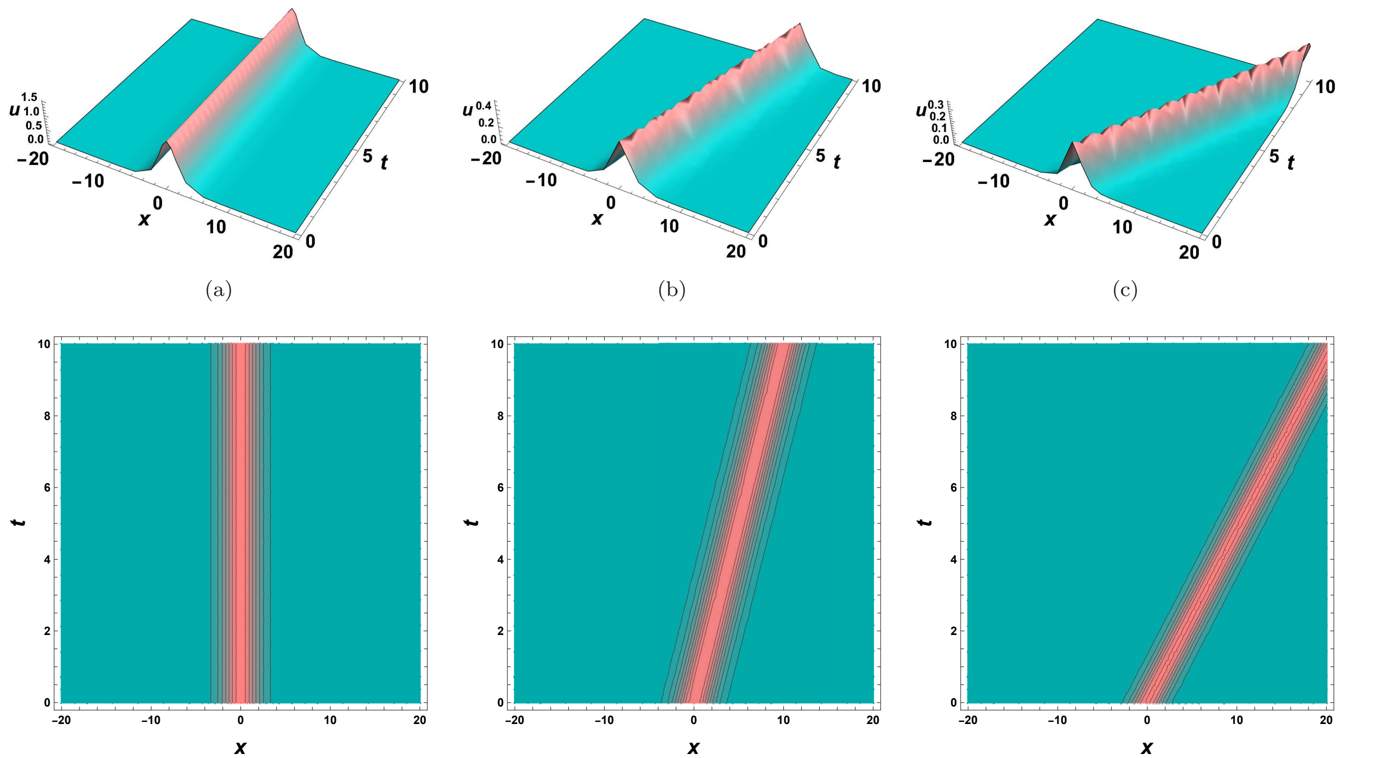
Deducing the terms τ_x and τ_{xx} from the equation (14) as

$$\tau_x = \delta_1 \mathrm{e}^{\left(\delta_1 x - \frac{1}{2}t\left(\delta_1^5 + \sqrt{4\zeta\delta_1^6 s + 4\delta_1^2 s^2 + \delta_1^{10}}\right)\right)}, \tag{15}$$

$$\tau_{xx} = \delta_1^2 \mathrm{e}^{\left(\delta_1 x - \frac{1}{2}t\left(\delta_1^5 + \sqrt{4\zeta\delta_1^6 s + 4\delta_1^2 s^2 + \delta_1^{10}}\right)\right)}. \tag{16}$$

On substituting the expressions from the Eqs. (14), (15) and (16) into Eq. (13), we obtain a single-soliton solution as

$$u(x,t) = -\frac{(21C_1 - 43C_2 + 32C_3)\delta_1^2 \mathrm{e}^{\left(\frac{1}{2}t\left(\delta_1^5 + \sqrt{\delta_1^2\left(4\zeta\delta_1^4 s + \delta_1^8 + 4s^2\right)}\right) + \delta_1 x\right)}}{30\left(\mathrm{e}^{\left(\frac{1}{2}t\left(\delta_1^5 + \sqrt{\delta_1^2\left(4\zeta\delta_1^4 s + \delta_1^8 + 4s^2\right)}\right)\right)} + \mathrm{e}^{\delta_1 x}\right)^2}. \tag{17}$$



(a) (b) (c)

(d) (e) (f)

Figure 1: Profiles for a single soliton via (13) with function (14) and the values ($\zeta = 1, s = 0$): **(a)** tmSK equation: $\delta_1 = 1, C_1 = \frac{5}{3}, C_2 = 5, C_3 = 0$; **(b)** tmLax equation: $\delta_1 = -1, C_1 = 10, C_2 = 10, C_3 = 5$; and **(c)** tmCDG equation: $\delta_1 = -1.2, C_1 = 60, C_2 = 30, C_3 = 0$. **(d-f)** plots the contours with respect to **(a-c)** in xt-plane.

3.2. Solution for two solitons

For a solution of two solitons, we take the τ as

$$\tau(x,t) = 1 + \mathrm{e}^{\Theta_1} + \mathrm{e}^{\Theta_2} + \chi_{12}\mathrm{e}^{\Theta_1+\Theta_2}, \tag{18}$$

where χ_{12} is the coefficient of dispersion. It can be calculated by putting τ and its derivatives from Eq. (18) into Eq. (13). We utilized symbolic computation to obtain the values for χ_{12} with existence condition $\xi = \zeta = 1$ as per suggested by [1] as

$$\chi_{12} = \frac{(\delta_1 - \delta_2)^2(\delta_1^2 - \delta_1\delta_2 + \delta_2^2)}{(\delta_1 + \delta_2)^2(\delta_1^2 + \delta_1\delta_2 + \delta_2^2)},$$

$$\chi_{12} = \frac{(\delta_1 - \delta_2)^2}{(\delta_1 + \delta_2)^2},$$

$$\chi_{12} = \frac{(\delta_1 - \delta_2)^2(\delta_1^2 - \delta_1\delta_2 + \delta_2^2)}{(\delta_1 + \delta_2)^2(\delta_1^2 + \delta_1\delta_2 + \delta_2^2)},$$

for tmSK, tmLax, and tmCDG, respectively. We create a system for $\{C_1, C_2, C_3\}$ with the variables $\{q_1, q_2, q_3\}$ to be determined correcponding to χ_{12} as

$$\begin{aligned} \frac{5}{3}q_1 + 5q_2 &= \frac{(\delta_1-\delta_2)^2(\delta_1^2-\delta_1\delta_2+\delta_2^2)}{(\delta_1+\delta_2)^2(\delta_1^2+\delta_1\delta_2+\delta_2^2)}, \\ 10q_1 + 10q_2 + 5q_3 &= \frac{(\delta_1-\delta_2)^2}{(\delta_1+\delta_2)^2}, \\ 60q_1 + 30q_2 &= \frac{(\delta_1-\delta_2)^2(\delta_1^2-\delta_1\delta_2+\delta_2^2)}{(\delta_1+\delta_2)^2(\delta_1^2+\delta_1\delta_2+\delta_2^2)}. \end{aligned} \tag{19}$$

We get the values for q_1, q_2, and q_3 by solving the system (19), given as

$$\begin{aligned} q_1 &= -\frac{1}{10}\frac{(\delta_1-\delta_2)^2(\delta_1^2-\delta_1\delta_2+\delta_2^2)}{(\delta_1+\delta_2)^2(\delta_1^2+\delta_1\delta_2+\delta_2^2)}, \\ q_2 &= \frac{7}{30}\frac{(\delta_1-\delta_2)^2(\delta_1^2-\delta_1\delta_2+\delta_2^2)}{(\delta_1+\delta_2)^2(\delta_1^2+\delta_1\delta_2+\delta_2^2)}, \\ q_3 &= -\frac{1}{15}\frac{(\delta_1-\delta_2)^2(\delta_1^2-7\delta_1\delta_2+\delta_2^2)}{(\delta_1+\delta_2)^2(\delta_1^2+\delta_1\delta_2+\delta_2^2)}. \end{aligned} \tag{20}$$

Therefore, we constitute a general value for $\chi_{12} = q_1C_1 + q_2C_2 + q_3C_3$ as

$$\chi_{12} = -\frac{(\delta_1-\delta_2)^2\{(3C_1-7C_2+2C_3)\delta_1^2-(3C_1-7C_2+14C_3)\delta_1\delta_2+(3C_1-7C_2+2C_3)\delta_2^2\}}{30(\delta_1+\delta_2)^2(\delta_1^2+\delta_1\delta_2+\delta_2^2)},$$

and it can be extended in general for the function τ

$$\tau(x,t) = 1 + \mathrm{e}^{\Theta_i} + \mathrm{e}^{\Theta_j} + \chi_{ij}\mathrm{e}^{\Theta_i+\Theta_j}, \tag{21}$$

as

$$\chi_{ij} = -\frac{(\delta_i-\delta_j)^2\{(3C_1-7C_2+2C_3)\delta_i^2-(3C_1-7C_2+14C_3)\delta_i\delta_j+(3C_1-7C_2+2C_3)\delta_j^2\}}{30(\delta_i+\delta_j)^2(\delta_i^2+\delta_i\delta_j+\delta_j^2)}, \quad 1 \le i < j \le N, \tag{22}$$

where N is a positive integer. Therefore, by Eq. (18) we get

$$\tau_x = \delta_1\mathrm{e}^{\Theta_1} + \delta_2\mathrm{e}^{\Theta_1} + \chi_{12}(\delta_1+\delta_2)\mathrm{e}^{\Theta_1+\Theta_2}, \tag{23}$$

$$\tau_{xx} = \delta_1^2\mathrm{e}^{\Theta_1} + \delta_2^2\mathrm{e}^{\Theta_1} + \chi_{12}(\delta_1+\delta_2)^2\mathrm{e}^{\Theta_1+\Theta_2}. \tag{24}$$

By substituting the Eqs. (18), (23) and (24) into Eq. (13), we obtain a two-soliton solution.

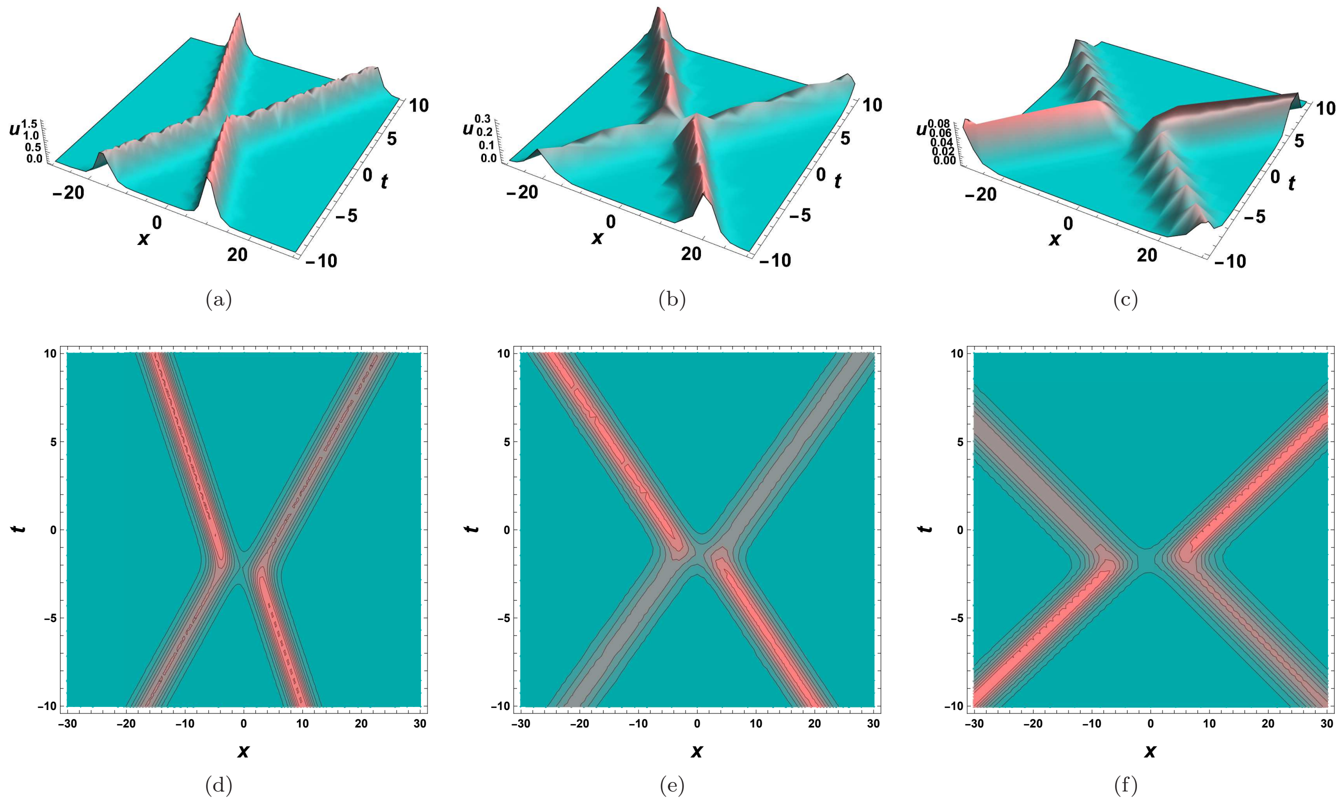


Figure 2: Profiles for the interaction of two solitons via (13) with function (18) and the values: **(a)** tmSK equation: $\delta_1 = 1.1, \delta_2 = -0.9, C_1 = \frac{5}{3}, C_2 = 5, C_3 = 0, \zeta = 1, s = 1$; **(b)** tmLax equation: $\delta_1 = -0.6, \delta_2 = 0.8, C_1 = 10, C_2 = 10, C_3 = 5, \zeta = 1, s = 2$; and **(c)** tmCDG equation: $\delta_1 = 0.5, \delta_2 = -0.6, C_1 = 60, C_2 = 30, C_3 = 0, \zeta = 1, s = 3$. **(d-f)** plots the contours with respect to **(a-c)** in xt-plane.

3.3. Solution for three solitons

To achieve a solution for three solitons [1, 44], we pressume the τ as

$$\tau(x,t) = 1 + \mathrm{e}^{\Theta_1} + \mathrm{e}^{\Theta_2} + \mathrm{e}^{\Theta_3} + \chi_{12}\mathrm{e}^{\Theta_1+\Theta_2} + \chi_{13}\mathrm{e}^{\Theta_1+\Theta_3} + \chi_{23}\mathrm{e}^{\Theta_2+\Theta_3} + \chi_{123}\mathrm{e}^{\Theta_1+\Theta_2+\Theta_3}, \tag{25}$$

where χ_{mn} with $1 \leq m < n \leq 3$ satisfies the Eq. (22) and χ_{123} is the coefficient of dispersion. The value of χ_{123} can be obtained using symbolic software to fulfill the constraint:

$$\chi_{123} = \chi_{12}\chi_{13}\chi_{23}. \tag{26}$$

Thus, from Eq. (25), we have

$$\tau_x = \sum_{p=1}^{3} \delta_p \mathrm{e}^{\Theta_p} + \sum_{1\leq p<q\leq 3} (\delta_p + \delta_q)\chi_{pq}\mathrm{e}^{\Theta_p+\Theta_q} + \left(\sum_{p=1}^{3} \delta_p\right) \chi_{123}\mathrm{e}^{\Theta_1+\Theta_2+\Theta_3}, \tag{27}$$

$$\tau_{xx} = \sum_{p=1}^{3} \delta_p^2 \mathrm{e}^{\Theta_p} + \sum_{1\leq p<q\leq 3} (\delta_p + \delta_q)^2\chi_{pq}\mathrm{e}^{\Theta_p+\Theta_q} + \left(\sum_{p=1}^{3} \delta_p\right)^2 \chi_{123}\mathrm{e}^{\Theta_1+\Theta_2+\Theta_3}. \tag{28}$$

On putting the Eqs. (25), (27) and (28) into Eq. (13), we get a solution for three solitons.

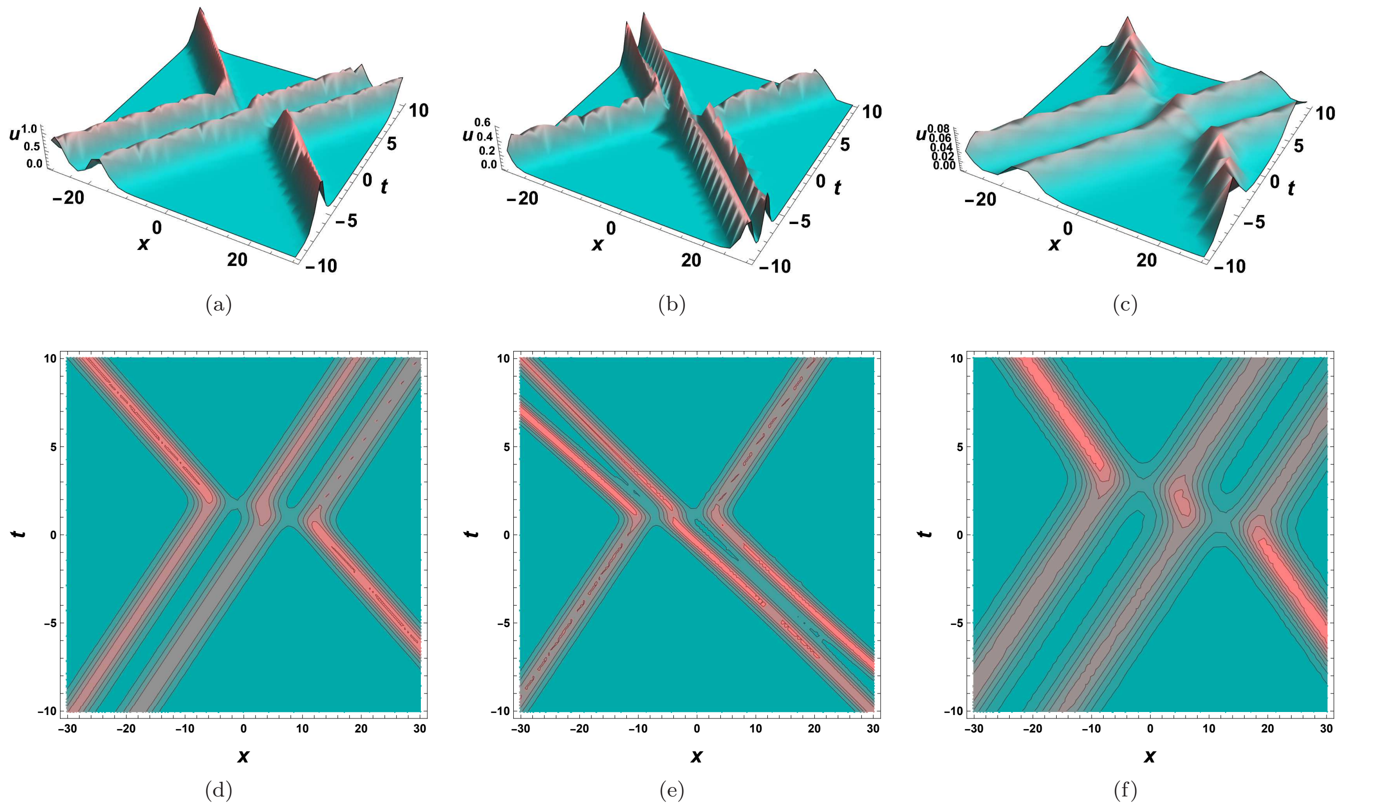


Figure 3: Profiles for the interaction of three solitons via (13) with function (25) and the values ($\zeta = 1, s = 2$): **(a)** tmSK equation: $\delta_1 = 0.6, \delta_2 = 0.7, \delta_3 = -0.9, C_1 = \frac{5}{3}, C_2 = 5, C_3 = 0$; **(b)** tmLax equation: $\delta_1 = -1, \delta_2 = -1.1, \delta_3 = 0.9, C_1 = 10, C_2 = 10, C_3 = 5$; and **(c)** tmCDG equation: $\delta_1 = -0.6, \delta_2 = 0.5, \delta_3 = 0.45, C_1 = 60, C_2 = 30, C_3 = 0$. **(d-f)** plots the contours with respect to **(a-c)** in xt-plane.

4. Lump and its interection with solitons

4.1. One lump solution

To establish a solution for lump, we pick the function τ in such a way that solution u comes to be a rational solution, as

$$\begin{aligned} \tau(x,t) &= X^2 + Y^2 + h_7, \\ X(x,t) &= h_1 x + h_2 t + h_3, \\ Y(x,t) &= h_4 x + h_5 t + h_6, \end{aligned} \tag{29}$$

where $h_i; 1 \leq i \leq 7$ are the constants that will be determined. On transforming the equation (5) with dependent variable transformation (13), we get a heptalinear or septalinear equation as

$$\begin{aligned} &12600 C_1 s^2 \tau_x^3 \tau^4 + 6300 C_1 \tau_{tt} \tau_x \tau^5 - 12600 C_1 \tau_t^2 \tau_x \tau^4 - 25800 C_2 s^2 \tau_x^3 \tau^4 - 12900 C_2 \tau_{tt} \tau_x \tau^5 + 25800 C_2 \tau_t^2 \tau_x \tau^4 \\ &+ 9600 C_3 \tau_{tt} \tau_x \tau^5 + \ldots + 12900 C_2 \tau_{(6x)t} \tau^6 + 9600 \zeta C_3 s \tau_{(7x)} \tau^6 - 67200 \zeta C_3 s \tau_x \tau_{(6x)} \tau^5 - 9600 C_3 \tau_{(6x)t} \tau^6 = 0, \end{aligned} \tag{30}$$

where midterms are skipped due to lengthy-expression. Using the symbolic computation, we substitute the equation (29) in the heptalinear equation (30) and collect the coefficients of independent variables

x and t. Therefore, we obtain a set of equations in the parameters $h_i; 1 \leq i \leq 7$. By solving these equations, one can get a set of solutions with some appropriate choices; we show here three sets of solutions as given below:
(i) $h_3 = h_6 = h_7 = 0, h_1 = \frac{-h_4 h_5}{h_2}; h_2 \neq 0, h_2 = h_2, h_4 = h_4, h_5 = h_5,$
(ii) $h_1 = h_5 = h_6 = 0, h_7 = -h_3^2, h_2 = h_2, h_3 = h_3, h_4 = h_4,$
(iii) $h_1 = h_5 = h_6 = 0, h_7 = \frac{3h_3^2}{5} h_2 = h_2, h_3 = h_3, h_4 = h_4.$
Thus, we get a one-lump solution by substituting Eq. (29) with a solution set into the Eq. (13) as

$$u(x,t) = \frac{(21C_1 - 43C_2 + 32C_3)h_4^2\left(h_4^2 x^2 - h_2^2 t^2\right)}{15\left(h_2^2 t^2 + h_4^2 x^2\right)^2}, \tag{31}$$

$$u(x,t) = \frac{(21C_1 - 43C_2 + 32C_3)h_4^2\left(h_2^2\left(-t^2\right) - 2h_3 h_2 t + h_4^2 x^2\right)}{15\left(h_2^2 t^2 + 2h_3 h_2 t + h_4^2 x^2\right)^2}, \tag{32}$$

$$u(x,t) = \frac{-(21C_1 - 43C_2 + 32C_3)h_4^2\left((h_2 t + h_3)^2 - h_4^2 x^2 + \frac{3h_3^2}{5}\right)}{15\left((h_2 t + h_3)^2 + h_4^2 x^2 + \frac{3h_3^2}{5}\right)^2}, \tag{33}$$

for the solution set (i), (ii) and (iii), respectively.

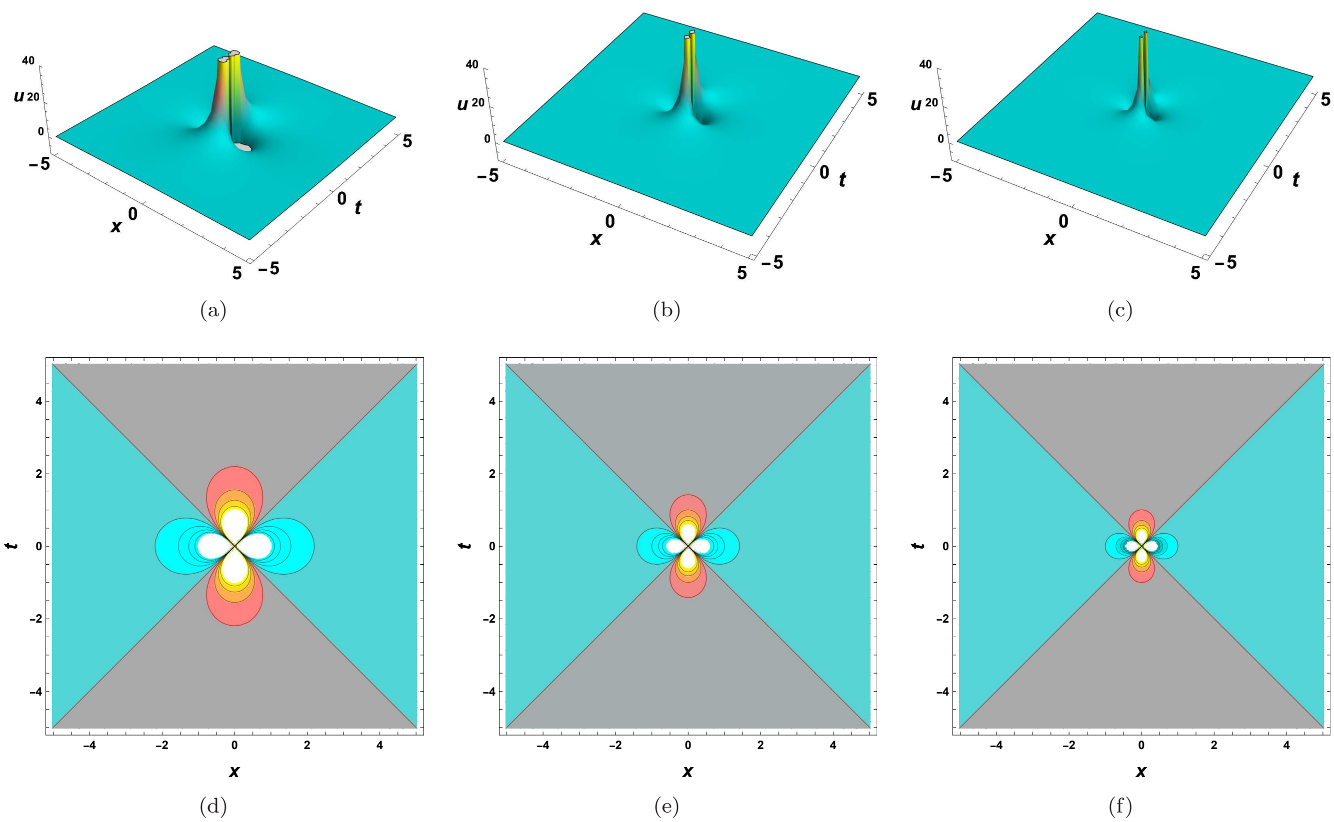
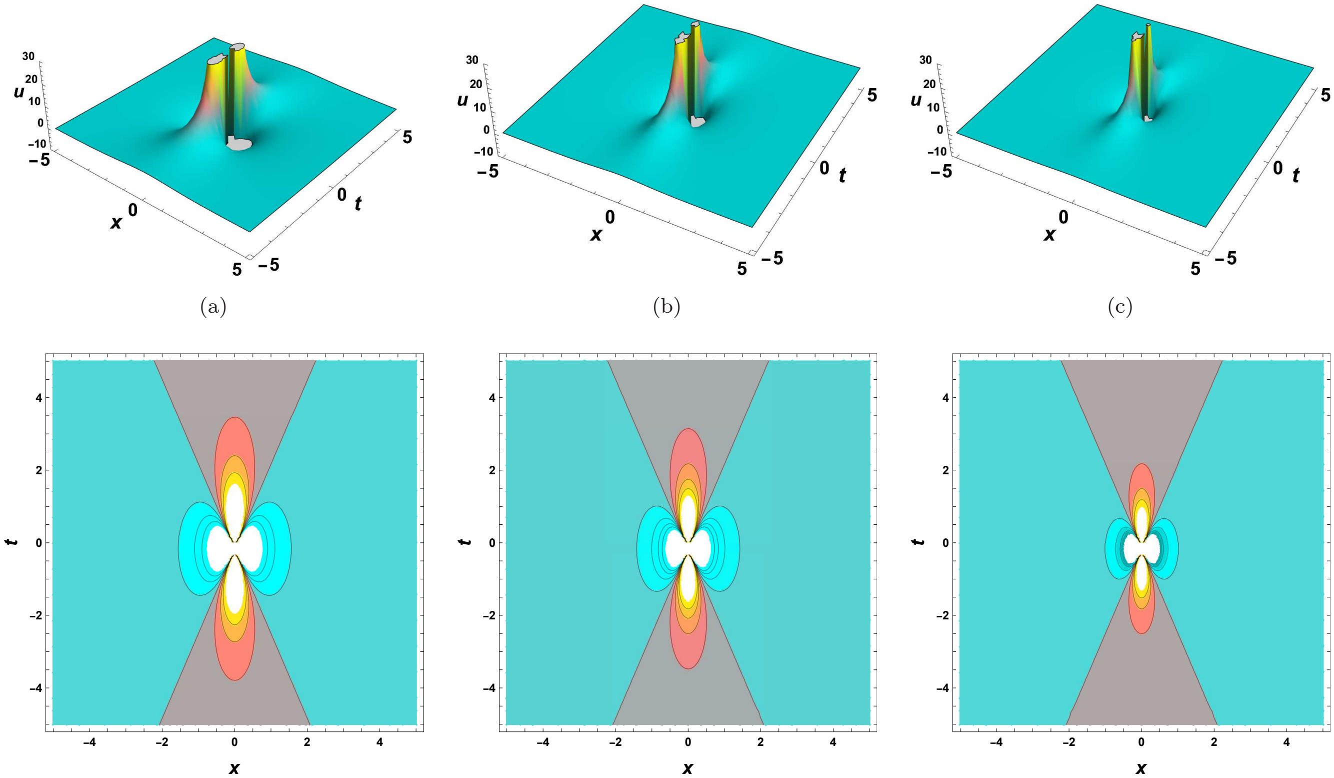


Figure 4: Profiles for one lump via (31) with function (29) and the values: $h_2 = 0.5, h_4 = 0.5$; **(a)** tmSK equation: $C_1 = \frac{5}{3}, C_2 = 5, C_3 = 0$; **(b)** tmLax equation: $C_1 = 10, C_2 = 10, C_3 = 5$; and **(c)** tmCDG equation: $C_1 = 60, C_2 = 30, C_3 = 0$. **(d-f)** plots the contours with respect to **(a-c)** in xt-plane.



(a) (b) (c)

(d) (e) (f)

Figure 5: Profiles for one lump via (32) with function (29) and the values: $h_2 = 3, h_3 = 0.5, h_4 = 5$; **(a)** tmSK equation: $C_1 = \frac{5}{3}, C_2 = 5, C_3 = 0$; **(b)** For tmLax equation: $C_1 = 10, C_2 = 10, C_3 = 5$; and **(c)** tmCDG equation: $C_1 = 60, C_2 = 30, C_3 = 0$. **(d-f)** plots the contours with respect to **(a-c)** in xt-plane.

4.2. One lump interaction with single soliton

In order to construct an interaction solution of a lump and a soliton, we assume the auxiliary function τ as

$$\begin{aligned} \tau(x,t) &= X^2 + Y^2 + h_7 + \mathrm{e}^{k_1 x + k_2 t}, \\ X(x,t) &= h_1 x + h_2 t + h_3, \\ Y(x,t) &= h_4 x + h_5 t + h_6, \end{aligned} \tag{34}$$

where $h_i; 1 \leq i \leq 7$ and $k_i; 1 \leq i \leq 2$ are the constants that will be determined. Using the symbolic computation, we substitute the equation (34) in the heptalinear equation (30) and collect the coefficients of independent variables x and t. Therefore, we obtain a set of equations in the parameters $h_i; 1 \leq i \leq 7$ and $k_i; 1 \leq i \leq 2$. On solving these equations, one can get a set of solutions with some appropriate choices, we show here a solution as given below:
$h_2 = \frac{1}{30}(441C_1^3h_1^5 - 1806C_1^2C_2h_1^5 + 1344C_1^2C_3h_1^5 + 1849C_1C_2^2h_1^5 - 7998C_2^2h_1^5 - 2752C_1C_2C_3h_1^5 - 240C_2C_3h_1^5 + 3906C_1C_2h_1^5 + 1024C_1C_3^2h_1^5 + 4608C_3^2h_1^5 + 3024C_1C_3h_1^5 + 113400h_1^5), h_3 = h_6 = h_7 = h_4 = 0, h_1 = h_1, h_5 = h_5, k_1 = k_1, k_2 = k_2$
Therefore, by substituting Eq. (34) with the solution set into the Eq. (13), we get an interaction solution of a lump with a single soliton.

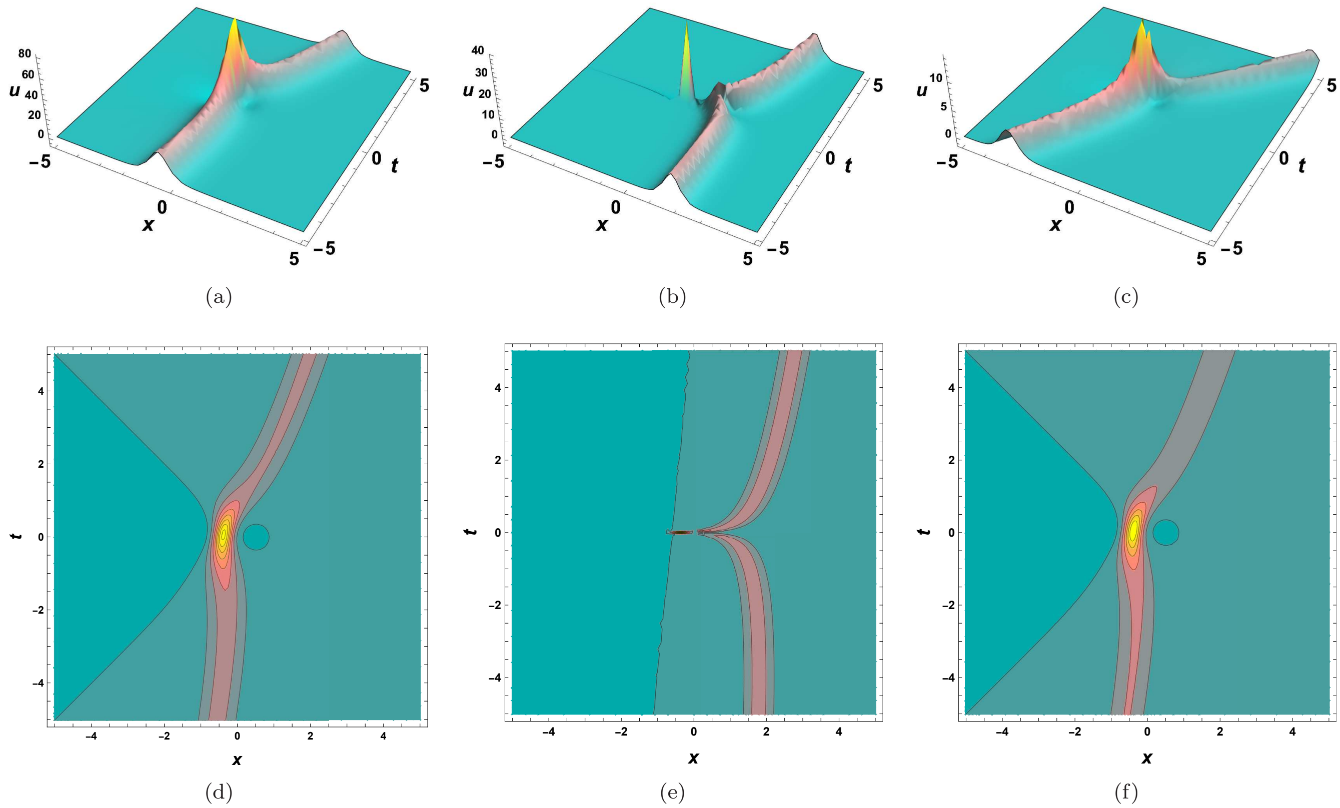


Figure 6: Profiles for the interaction of one lump and a single soliton via (13) with function (34) and the values: $h_1 = 0.8, h_5 = 0.8, k_1 = 4, k_2 = 1$; **(a)** tmSK equation: $C_1 = \frac{5}{3}, C_2 = 5, C_3 = 0$; **(b)** tmLax equation: $C_1 = 10, C_2 = 10, C_3 = 5$; and **(c)** tmCDG equation: $C_1 = 60, C_2 = 30, C_3 = 0$. **(d-f)** plots the contours with respect to **(a-c)** in xt-plane.

5. Results and discussion

It is well known that the KdV equations of fifth-order, such as Lax, CDG, and SK equations, are completely integrable. As the tmLax, tmCDG, and tmSK are the two-mode equations for Lax, CDG, and SK equations, respectively; therefore, all convey the same integrable properties and multiple solitons depending on the parameters C_1, C_2, C_3, s, ξ, and ζ. We obtained soliton, lump, and interaction solutions for the said equations and showcased the dynamics for the obtained solutions. Thus, the results can be discussed as

- In figure 1, we illustrate the single soliton solution due to the movable singularity at $x = 0$ with respect to the parameters $\delta_1 = 1, \zeta = 1, s = 0$, $\delta_1 = -1.2, \zeta = 1, s = 0$ and $\delta_1 = -1.2, \zeta = 1, s = 0$ for tmSK, tmLax and tmCDG, respectively.

- Figure 2 shows the interaction between two solitons with their corresponding movable singularities with different scale parameter $s = 1, 2$, and 3. We obtain the interaction of two solitons for the parameters $\delta_1 = 1.1, \delta_2 = -0.9, \zeta = 1, s = 1$, $\delta_1 = -0.6, \delta_2 = 0.8, \zeta = 1, s = 2$ and $\delta_1 = 0.5, \delta_2 = -0.6, \zeta = 1, s = 3$, for tmSK, tmLax and tmCDG, respectively.

- In figure 3, we depict the three solitons interaction with respect to the movable singularities on fixed scale parameter $s = 2$. We get three solitons interaction for the parameters $\delta_1 = 1.1, \delta_2 = -0.9, \zeta = 1, s = 1$, $\delta_1 = -0.6, \delta_2 = 0.8, \zeta = 1, s = 2$ and $\delta_1 = 0.5, \delta_2 = -0.6, \zeta = 1, s = 3$, for tmSK, tmLax and tmCDG, respectively.

- Figure 4 illustrate the dynamics of single lump solution due to the singularity at $x = -\frac{h_2}{h_4}t; h_4 \neq 0$ with fixed values of parameters $h_2 = h_4 = 0.5$ for the tmSK, tmLax, and tmCDG equations.

- In figure 5, we show the depiction of single lump solution with singularity at $x = \frac{\sqrt{h_2^2 t^2 + 2h_3 h_2 t}}{h_4}; h_4 \neq 0$ with respect to the fixed parameters $h_2 = 3, h_3 = 0.5, h_4 = 5$ for the tmSK, tmLax, and tmCDG equations.

- Figure 6 depicts the interaction between one lump and a single soliton for the fixed parameters $h_1 = h_5 = 0.8, k_1 = 4, k_2 = 1$ for the tmSK, tmLax and tmCDG equations.

6. Conclusions

In summary, we investigated a generalized nonlinear two-mode fifth-order partial differential equation and studied it for tmSK, tmLax, and tmCDG equations. We examined this generalized two-mode evolution equation by a logarithmic transformation concerning the dispersion of the equation. We used the simplified technique of the Hirota method to get the solutions for multiple solitons and constructed one, two, and three solitons and the interactions. Also, We built a one-lump solution and its interaction with a soliton. We showcased the dynamics of the obtained solutions through 3D graphics with their contour plots by choosing appropriate values for the parameters. We utilized the symbolic computational software *Mathematica* to obtain the desired parameters and the solutions for multi-solitons, lump, and their interactions. The tmSK, tmLax, and tmCDG equations simultaneously study the propagation of two-mode waves in the identical direction with different phase velocities, dispersion parameters, and nonlinearity. These equations have applications in several real-life examples, such as gravity-affected waves or gravity-capillary waves, waves in shallow water, propagating waves in fast-mode and the slow-mode with their phase velocity in a strong and weak magnetic field, known as magneto-sound propagation in plasmas.

In the extension work, our focus is on obtaining the different solutions other than the obtained solutions, such as breathers and their interactions with solitons and lumps. Also, anyone will be able to apply the approach methodology to get a general logarithmic transformation for generalized nonlinear equations with different values of R in its dependent variable transformation. Another future work direction is to seek the solutions for different temporal dispersions, such as third-order or more, and analyze their impact or changes concerning the standard known nonlinear equations.

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Data availability statement

The authors traced the dynamics of various solitons and lumps with *Mathematica*. There is no data taken from outside sources.

Conflict of interest

The authors declare that they have no conflict of interest.

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