Rogue-wave structures for a generalized (3+1)-dimensional nonlinear wave equation in liquid with gas bubbles

Brij Mohan*[∗] and Sachin Kumar[†]

[∗]Department of Mathematics, Hansraj College, University of Delhi, Delhi-110007, India

†Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi-110007, India

Abstract: This study explores the behavior of higher-order rogue waves within a (3+1)-dimensional generalized nonlinear wave equation in liquid-containing gas bubbles. It creates the investigated equation's Hirota D-operator bilinear form. We employ a generalized formula with real parameters to obtain the rogue waves up to the third order using the direct symbolic technique. The analysis reveals that the second and third-order rogue solutions produce two and three-waves, respectively. To gain deeper insights, we use the Cole-Hopf transformation on the transformed variables ξ and η to produce a bilinear equation. Using system software *Mathematica*, the dynamic analysis presents the graphics for the obtained solutions in transformed ξ, η and original spatial-temporal coordinates x, y, z, t. These visualizations reveal rogue waves' intricate structure and evolution, capturing their localized interactions and significant presence within nonlinear systems. We demonstrate that rogue waves, characterized by their substantial height and sudden appearance, are prevalent in various nonlinear events. The equation examined in this study offers valuable insights into the evolution of longer waves with smaller amplitudes, which is particularly relevant in fields such as fluid dynamics, dispersive media, and plasmas. The implications of this research extend across multiple scientific domains, including fiber optics, oceanography, dusty plasma, and nonlinear systems, where understanding the behavior of rogue waves is crucial for both theoretical and practical applications.

Keywords: Cole-Hopf transformation; Bilinear form; Dispersion; Direct symbolic approach; Higher-order waves.

1 Introduction

Partial differential equations (PDEs) [1–9] containing dependent variable functions and their partial derivatives are a significant topic in applied mathematics and mathematical physics. Several nonlinear sciences and engineering fields employ PDEs to represent complex physical procedures. Mathematicians have utilized nonlinear PDEs to explain various scientific phenomena, including gravitational research and fluid dynamics. Analyzing and solving nonlinear PDEs can be challenging because no universal method exists. In many different nonlinear sciences, PDEs represent and comprehend physical phenomena that contain numerous variables and their derivatives. The wave equation [10], heat equation [11], and the well-known Schrödinger equation [12] from quantum mechanics are a few examples of PDEs. The Bäcklund transformation [3, 4], Hirota's bilinearization method [5–7], Darboux transformation [8, 9], inverse scattering method [13, 14], bilinear neural network method [15, 16], simplified Hirota's approach [17, 18], Lie symmetry approach [19–21], and other techniques are used to solve nonlinear evolution equations and obtain the analytical and exact solutions.

[∗] *brijmohan6414@gmail.com*

[†] *sachinambariya@gmail.com (Corresponding author)*

Rogue waves, sometimes known as extreme waves [22–32], are large scaled localized waves in space and time. They threaten sea-farers, ships and vessels, and other entities. However, it may also help extract useful information about a system and its behavior in non-oceanic cases. For example, it can lead to the formation of extreme wave localization in optics. The evolution of rogue waves is an important topic for many scholars. Unlike typical ocean waves, rogue waves can reach towering heights of 20–30 meters or more, often appearing unexpectedly in relatively calm seas or during storms, making them extremely dangerous for ships and offshore structures. The height of rogue waves distinguishes them from neighboring waves. Rogue waves are investigated in nonlinear wave dynamics because they violate widely accepted linear wave theories. The goal of scientific study on exceptional or large waves is to predict their occurrence and comprehend the fundamental principles that underlie them. When shorter waves concentrate their energy on a narrow area, rogue waves occurs from nowhere. The sea safety is one such improvement use. In order to prevent the harm that these waves can impose, the evolving models and algorithms give early warning and detections. The marine sector, the coastal region, and offshore oil and gas areas could all benefit from knowing this information. Thus, knowing the mechanics of rogue waves helps develop safe structures and mitigation techniques. Thus, getting operational safety and reasonable solutions are feasible. Furthermore, examining the dynamics and origins adds to our understanding of complicated procedures and the formation of extreme processes in various nonlinear sciences.

In this work, we examine newly constructed rogue waves of a generalized (3+1)-dimensional nonlinear wave equation [33, 34] in liquid with gas bubbles as

$$(u_t + \alpha_1 u u_x + \alpha_2 u_{xxx} + \alpha_3 u_x)_x + \alpha_4 u_{yy} + \alpha_5 u_{zz} = 0, \tag{1}$$

where $u(x, y, z, t)$ is a wave function representing the amplitude of the propagating wave as a function of space and time, and $\alpha_{1 \le i \le 5}$ are non-zero constants. The nonlinear term $\alpha_1 u u_x$ represents the self-interaction of the wave, which captures the formation of rogue waves, vast and unexpected waves that occur in the nonlinear system. The term $\alpha_2 u_{xxx}$ accounts for the dispersive effects in the medium, where different waves travel at different speeds, conducting the spreading of the wave packet. The dissipative term $\alpha_3 u_x$ represents the energy loss or dissipation in the medium due to factors like viscosity in the liquid and the cross-diffusion terms $\alpha_4 u_{yy}$ and $\alpha_5 u_{zz}$ are a diffusion of the wave in the transverse directions y and z describing the nature of wave propagation.

This research concentrates on originating rogue wave solutions to this generalized nonlinear evolution equation and investigating their dynamics. The rogue waves in this model emphasize the possibility for sudden, large amplitude waves in the liquid medium with gas bubbles. This is relevant in comprehending underwater explosions, sonic booms, or extreme oceanic rogue waves. By exploring the dynamics of these rogue waves, the study provides wisdom into how such waves form, evolve, and dissipate over time. This learning can potentially guide the development of processes to predict and mitigate the consequences of rogue waves in real-world scenarios, offering hope in the face of these unforeseen natural phenomena. The relevance of this research to understanding and predicting rogue waves in various scenarios keeps the researchers engaged and interested in the topic. Overall, the physical relevance of this model lies in its ability to capture complex wave phenomena in a nonlinear medium with multiple interacting effects (nonlinearity, dispersion, dissipation, and cross-diffusion), providing a more profound understanding of the mechanisms after rogue wave appearance and propagation.

The equation (1) models the propagation of waves in a liquid medium containing gas bubbles. This equation accounts for the complex behavior of waves as they interact with the bubbles within the liquid, capturing the effects of non-linearity, dispersion, and scattering in a three-dimensional space over time. Such modeling is crucial in various physical and engineering applications, including underwater acoustics, bio-medical ultrasound, and industrial processes involving cavitation. The exact solutions and symmetry reductions explored in the studies [33, 34] offer valuable insights into the fundamental dynamics of these wave phenomena, making the equation a vital tool for predicting and understanding how waves behave in bubbly liquids. The associated conservation laws also ensure that the model adheres to essential physical principles, such as energy conservation and momentum, further validating its applicability in real-world scenarios. This equation generalizes the well-known equations from different areas of nonlinear science as

- (3+1)-dimensional nonlinear wave equation [35] for $\alpha_1 = 1, \alpha_2 = \frac{1}{4}, \alpha_3 = -1$, and $\alpha_4 = \alpha_5 = \frac{3}{4}$ as

$$(u_t + uu_x + \frac{1}{4}u_{xxx} - u_x)_x + \frac{3}{4}\left(u_{yy} + u_{zz}\right) = 0, \tag{2}$$

- (3+1)-dimensional Kadomtsev–Petviashvili equation [36] for $\alpha_1 = -6, \alpha_2 = 1, \alpha_3 = 0$, and $\alpha_4 = \alpha_5 = 3$ as

$$(u_t - 6uu_x + u_{xxx})_x + 3\left(u_{yy} + u_{zz}\right) = 0, \tag{3}$$

- (3+1)-dimensional nonlinear wave equation [37] for $\alpha_1 = \alpha_2 = 1, \alpha_3 = 0$, and $\alpha_4 = \alpha_5 = \frac{1}{2}$ as

$$(u_t + uu_x + u_{xxx})_x + \frac{1}{2}\left(u_{yy} + u_{zz}\right) = 0, \tag{4}$$

- (2+1)-dimensional Kadomtsev–Petviashvili equation [38] for $\alpha_1 = \alpha_2 = 1, \alpha_3 = 0, \alpha_4 = 1$, and $\alpha_5 = 0$ as

$$(u_t + uu_x + u_{xxx})_x + u_{yy} = 0, \tag{5}$$

- (1+1)-dimensional Korteweg–de Vries equation [39] for $\alpha_1 = \alpha_2 = 1$, and $\alpha_3 = \alpha_4 = \alpha_5 = 0$ as

$$u_t + uu_x + u_{xxx} = 0, \tag{6}$$

These equations (2)-(6), derived from various physical systems, often describe wave phenomena. The KdV-type equations are crucial in the field of plasma physics, as they provide insights into the behavior and structures of waves. Our study is significant as it constructs the bilinear form for the examined equation in transformed variables and uses a direct symbolic technique to visually evaluate the new rogue waves. This technique transforms the studied equation into a new (1+1)-dimensional evolution equation in the transformed variables. Importantly, we show that the investigated equation can be transformed into a bilinear form in the auxiliary function using the Cole-Hopf transformation, which has practical implications for understanding and predicting nonlinear wave behavior in plasmas.
Our study advances by deriving higher-order rogue wave solutions for investigating (3+1)-dimensional generalized nonlinear wave equation in liquid-containing gas bubbles using its Hirota bilinear form. While previous studies [33–38] have primarily focused on lower-order rogue waves or different nonlinear solutions, our research extends this understanding to more complex scenarios involving second and third-order rogue waves. This is particularly significant from a mathematical viewpoint, as it applies advanced techniques like the Cole-Hopf transformation and direct symbolic methods to obtain and analyze these solutions. From a physical perspective, our model's depiction of higher-order rogue waves is crucial as it reveals the intricate dynamics and interactions that occur in nonlinear systems, which are not captured by lower-order solutions. These higher-order rogue waves provide a more comprehensive understanding of the evolution of large waves from smaller amplitudes, which is essential for accurately modeling real-world phenomena in various scientific domains, including oceanography, dusty plasma, and fiber optics.

In this manuscript, the next section details the direct symbolic technique for identifying the solutions for rogue waves to the analyzed equation. It involves using the Cole-Hopf transformation in transformed variables to obtain a bilinear equation and determines the rogue waves up to the 3^{rd}-order alongwith their dynamics. Section 3 will present and discuss the results and findings, while the final section will conclude the research study.

2 Rogue waves via direct symbolic approach

We transform the equation (1) with $\xi = x + t$; $\eta = y + z$ in $u(x, y, z, t) = u(\xi, \eta)$ as

$$u_{\xi\xi} + \alpha_1(uu_{\xi\xi} + u_\xi^2) + \alpha_2 u_{\xi\xi\xi\xi} + \alpha_3 u_{\xi\xi} + \alpha_4 u_{\eta\eta} + \alpha_5 u_{\eta\eta} = 0. \tag{7}$$

Taking the phase $\theta_{i\in N}$ in equation (7) as

$$\theta_i = p_i\xi - w_i\eta, \tag{8}$$

having constants $p_{i\in N}$ and dispersions $w_{i\in N}$. In linear terms of Eq. (7), having $u(\xi, \eta) = e^{\theta_i}$ gives

$$w_i = \frac{\pm ip_i\sqrt{1 + \alpha_3 + \alpha_2 p_i^2}}{\sqrt{\alpha_4 + \alpha_5}}. \tag{9}$$

Considering the dependent variable transformation as

$$u(\xi, \eta) = K(\log f)_{\xi\xi}, \tag{10}$$

with nonzero constant K and auxiliary function $f(\xi, \eta)$. Substitution of equation (10) with $f = 1 + e^{\theta_1}$ in equation (7) gives

$$K = \frac{12\alpha_2}{\alpha_1}.$$

Thus, the equation (7) can be transformed using the transformation (10) into a bilinear equation in f as

$$\alpha_2(ff_{\xi\xi\xi\xi} - 4f_\xi f_{\xi\xi\xi} + 3f_{\xi\xi}^2) + (1 + \alpha_3)(ff_{\xi\xi} - f_\xi^2) + (\alpha_4 + \alpha_5)(ff_{\eta\eta} - f_\eta^2) = 0, \tag{11}$$

that further gives a bilinear form in Hirota D-operator [39] as

$$\left[\alpha_2 D_\xi^4 + (1 + \alpha_3)D_\xi^2 + (\alpha_4 + \alpha_5)D_\eta^2\right] f.f = 0, \tag{12}$$

where differential operators $D_{i=x,y}$ is defined as

$$D_x^{r_1} D_y^{r_2} U(x, y)V(x, y) = \left(\frac{\partial}{\partial_x} - \frac{\partial}{\partial_{x'}}\right)^{r_1} \left(\frac{\partial}{\partial_y} - \frac{\partial}{\partial_{y'}}\right)^{r_2} U(x, y)V(x', y')|_{x=x',y=y'},$$

with formal variables x', y' and positive integers $r_{i=1,2}$.
We construct the rogue waves by assuming the function f [40, 41] as

$$f(\xi, \eta) = \sum_{q=0}^{\frac{n(n+1)}{2}} \sum_{i=0}^{q} q_{n(n+1)-2q,2i}\xi^{n(n+1)-2q}\eta^{2i}, \tag{13}$$

where $s_{l,m\in\{0,2,\cdots,q(q+1)\}}$ are the real parameters.

2.1 First-order rogue waves

Having $n = 1$ in equation (13), we get f as

$$f(\xi, \eta) = f_1 = s_{2,0}\xi^2 + s_{0,2}\eta^2 + s_{0,0}. \tag{14}$$

We get a system of equations by putting the equation (14) in Eq. (11) with equating all coefficients of distinct powers of ξ and η to zero as

$$
\begin{aligned}
2\left(\alpha_4 s_{0,0} s_{0,2}+\alpha_5 s_{0,0} s_{0,2}+s_{2,0}\left(\left(\alpha_3+1\right) s_{0,0}+6\alpha_2 s_{2,0}\right)\right) &= 0,\\
2 s_{2,0}\left(\alpha_4 s_{0,2}+\alpha_5 s_{0,2}-\left(\alpha_3+1\right) s_{2,0}\right) &= 0,\\
2 s_{0,2}\left(-\alpha_4 s_{0,2}-\alpha_5 s_{0,2}+\left(\alpha_3+1\right) s_{2,0}\right) &= 0.
\end{aligned}
\tag{15}
$$

On solving the system gives parameter values as

$$
s_{0,0}=-\frac{3\alpha_2 s_{2,0}}{\alpha_3+1}, \qquad s_{0,2}=\frac{\alpha_3 s_{2,0}+s_{2,0}}{\alpha_4+\alpha_5}, \qquad s_{2,0}=s_{2,0}. \tag{16}
$$

So, the function (14) becomes

$$
f=f_1=s_{2,0}\left(\frac{\left(\alpha_3+1\right)\eta^2}{\alpha_4+\alpha_5}-\frac{3\alpha_2}{\alpha_3+1}+\xi^2\right), \tag{17}
$$

So, we get the solution by putting the equation (17) into (10) as

$$
u(\xi,\eta)=u_1=\frac{24\alpha_2\left(\frac{(\alpha_3+1)\eta^2}{\alpha_4+\alpha_5}-\frac{3\alpha_2}{\alpha_3+1}-\xi^2\right)}{\alpha_1\left(\frac{(\alpha_3+1)\eta^2}{\alpha_4+\alpha_5}-\frac{3\alpha_2}{\alpha_3+1}+\xi^2\right)^2}, \tag{18}
$$

2.2 Second-order rogue waves

Considering the function f for $n=2$ in equation (13) as

$$
f(\xi,\eta)=f_2=s_{6,0}\xi^6+s_{4,2}\eta^2\xi^4+s_{4,0}\xi^4+s_{2,4}\eta^4\xi^2+s_{2,2}\eta^2\xi^2+s_{2,0}\xi^2+s_{0,6}\eta^6+s_{0,4}\eta^4+s_{0,2}\eta^2+s_{0,0}. \tag{19}
$$

Substitution of Eq. (19) into (12) gives a system on equating to zero the coefficients of distinct powers of ξ and η. On solving, we get the parameters as

$$
\begin{gathered}
s_{0,0}=-\frac{625\alpha_2^3\left(\alpha_4+\alpha_5\right) s_{4,2}}{\left(\alpha_3+1\right)^4}, \qquad s_{0,2}=\frac{475\alpha_2^2 s_{4,2}}{3\left(\alpha_3+1\right)^2}, \qquad s_{0,4}=-\frac{17\alpha_2 s_{4,2}}{3\left(\alpha_4+\alpha_5\right)}, \qquad s_{0,6}=\frac{\left(\alpha_3+1\right)^2 s_{4,2}}{3\left(\alpha_4+\alpha_5\right)^2},\\
s_{2,0}=-\frac{125\alpha_2^2\left(\alpha_4+\alpha_5\right) s_{4,2}}{3\left(\alpha_3+1\right)^3}, \qquad s_{2,2}=-\frac{30\alpha_2 s_{4,2}}{\alpha_3+1}, \qquad s_{2,4}=\frac{\left(\alpha_3+1\right) s_{4,2}}{\alpha_4+\alpha_5}, \qquad s_{4,0}=-\frac{25\alpha_2\left(\alpha_4+\alpha_5\right) s_{4,2}}{3\left(\alpha_3+1\right)^2},\\
s_{6,0}=\frac{\left(\alpha_4+\alpha_5\right) s_{4,2}}{3\left(\alpha_3+1\right)},
\end{gathered}
\tag{20}
$$

Therefore, the function (19) becomes

$$
\begin{aligned}
f(\xi,\eta)=f_2=\frac{s_{4,2}}{3}\Bigg(&\frac{25\alpha_2^2\left(19\alpha_3\eta^2-5\alpha_4\xi^2-5\alpha_5\xi^2+19\eta^2\right)}{\left(\alpha_3+1\right)^3}+\frac{\left(\alpha_3\eta^2+\alpha_4\xi^2+\alpha_5\xi^2+\eta^2\right)^3}{\left(\alpha_3+1\right)\left(\alpha_4+\alpha_5\right)^2}-\\
&\alpha_2\left(\frac{17\eta^4}{\alpha_4+\alpha_5}+\frac{90\eta^2\xi^2}{\alpha_3+1}+\frac{25\alpha_4\xi^4}{\left(\alpha_3+1\right)^2}+\frac{25\alpha_5\xi^4}{\left(\alpha_3+1\right)^2}\right)-\frac{1875\left(\alpha_4+\alpha_5\right)\alpha_2^3}{\left(\alpha_3+1\right)^4}\Bigg),
\end{aligned}
\tag{21}
$$

which gives the solution on substituting it into (10)

$$
u(\xi,\eta)=u_2=\frac{12\alpha_2}{\alpha_1}(\log f_2)_{\xi\xi}. \tag{22}
$$

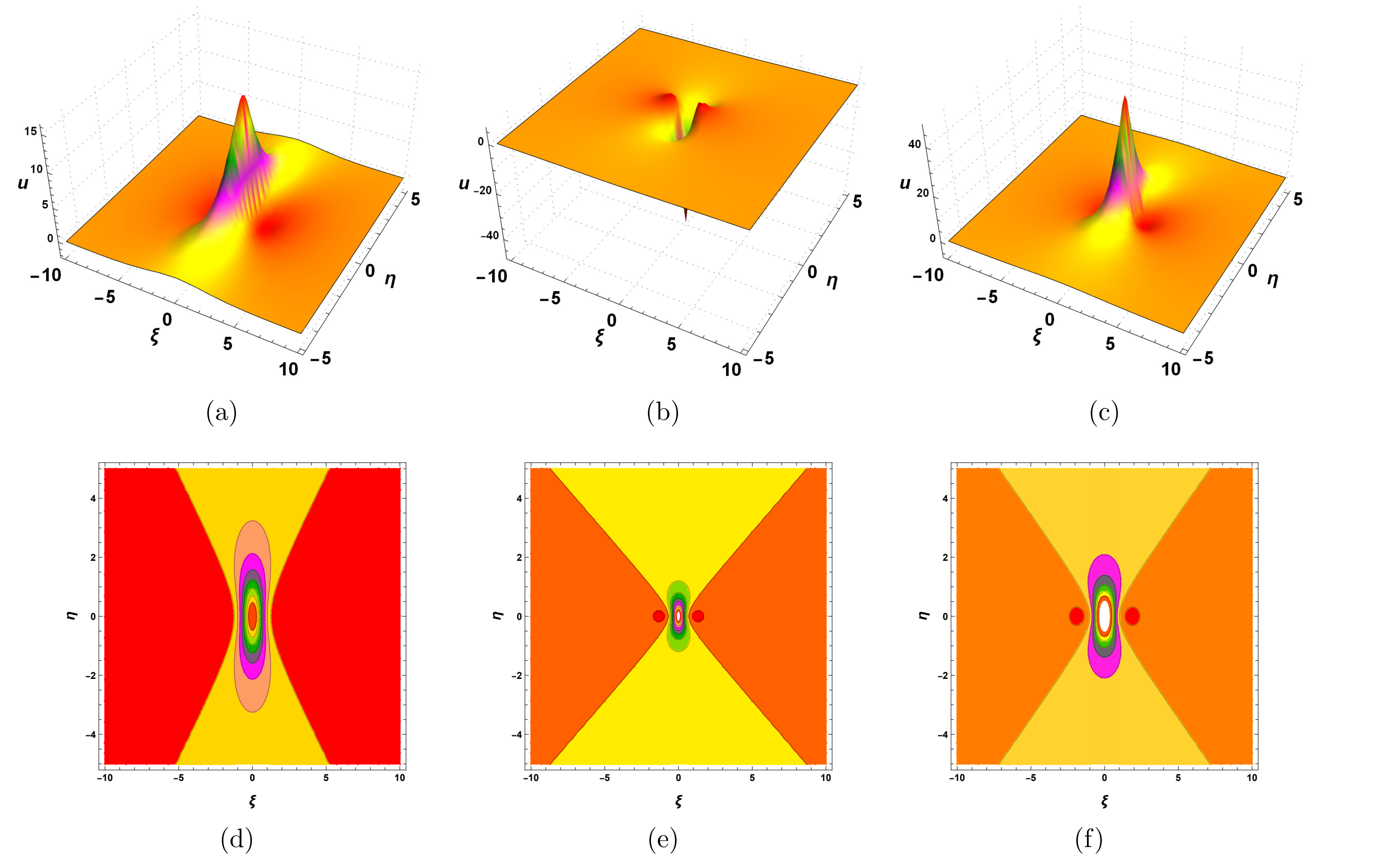


Figure 1: Dynamics of 1^{st}-order rogue waves for (18) in transformed variables ξ and η. **(d)-(f)** are the contours for **(a)-(c)** in $\xi\eta$-plane.

2.3 Third-order rogue waves

Having $n = 3$ in Eq. (13) gives the function f as

$$\begin{aligned} f(\xi,\eta) = f_3 = {} & s_{0,0} + s_{0,2}\eta^2 + s_{0,4}\eta^4 + s_{0,6}\eta^6 + s_{0,8}\eta^8 + s_{0,10}\eta^{10} + s_{0,12}\eta^{12} + s_{2,0}\xi^2 + s_{2,2}\xi^2\eta^2 + s_{2,4}\xi^2\eta^4 \\ & + s_{2,6}\xi^2\eta^6 + s_{2,8}\xi^2\eta^8 + s_{2,10}\xi^2\eta^{10} + s_{4,0}\xi^4 + q_{4,2}\xi^4\eta^2 + s_{4,4}\xi^4\eta^4 + s_{4,6}\xi^4\eta^6 + s_{4,8}\xi^4\eta^8 + s_{6,0}\xi^6 \\ & + s_{6,2}\xi^6\eta^2 + s_{6,4}\xi^6\eta^4 + s_{6,6}\xi^6\eta^6 + s_{8,0}\xi^8 + s_{8,2}\xi^8\eta^2 + s_{8,4}\xi^8\eta^4 + s_{10,0}\xi^{10} + s_{10,2}\xi^{10}\eta^2 + s_{12,0}\xi^{12}. \end{aligned} \tag{23}$$

On putting the Eq. (23) into (12), we get a system on equating to zero the coefficients of distinct powers of ξ and η. On solving the system, we get the parameters as

$$s_{0,0} = \frac{878826025\alpha_2^6(\alpha_4+\alpha_5)s_{10,2}}{54(\alpha_3+1)^7}, \quad s_{0,2} = -\frac{150448375\alpha_2^5 s_{10,2}}{9(\alpha_3+1)^5}, \quad s_{0,4} = \frac{16391725\alpha_2^4 s_{10,2}}{18(\alpha_3+1)^3(\alpha_4+\alpha_5)},$$

$$s_{0,6} = -\frac{399490\alpha_2^3 s_{10,2}}{9(\alpha_3+1)(\alpha_4+\alpha_5)^2}, \quad s_{0,8} = \frac{1445\alpha_2^2(\alpha_3+1)s_{10,2}}{2(\alpha_4+\alpha_5)^3}, \quad s_{0,10} = -\frac{29\alpha_2(\alpha_3+1)^3 s_{10,2}}{3(\alpha_4+\alpha_5)^4},$$

$$s_{0,12} = \frac{(\alpha_3+1)^5 s_{10,2}}{6(\alpha_4+\alpha_5)^5}, \quad s_{2,0} = -\frac{79893275\alpha_2^5(\alpha_4+\alpha_5)s_{10,2}}{9(\alpha_3+1)^6}, \quad s_{2,2} = \frac{94325\alpha_2^4 s_{10,2}}{(\alpha_3+1)^4},$$

$$s_{2,4} = \frac{2450\alpha_2^3 s_{10,2}}{(\alpha_3+1)^2(\alpha_4+\alpha_5)}, \quad s_{2,6} = \frac{17710\alpha_2^2 s_{10,2}}{3(\alpha_4+\alpha_5)^2}, \quad s_{2,8} = -\frac{95\alpha_2(\alpha_3+1)^2 s_{10,2}}{(\alpha_4+\alpha_5)^3},$$

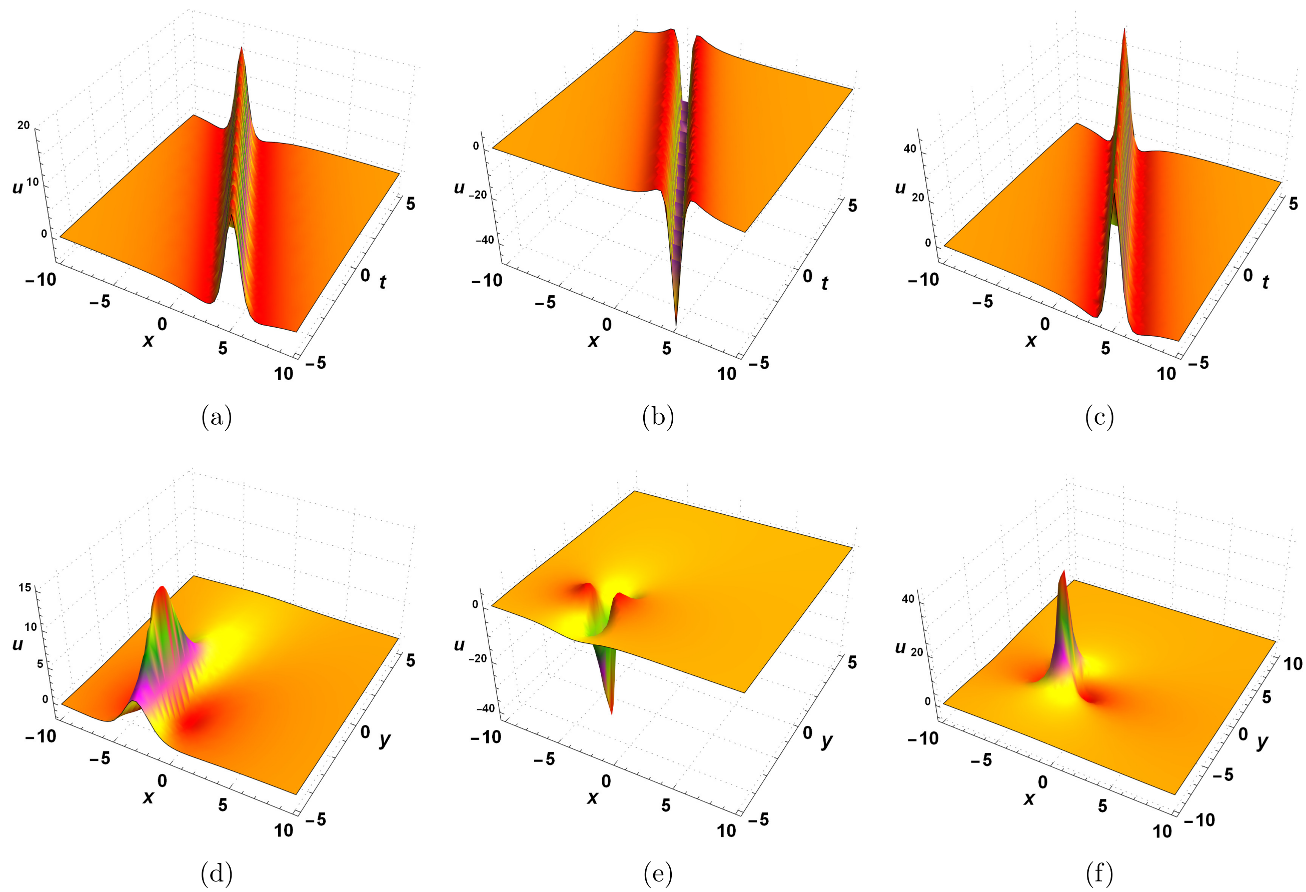
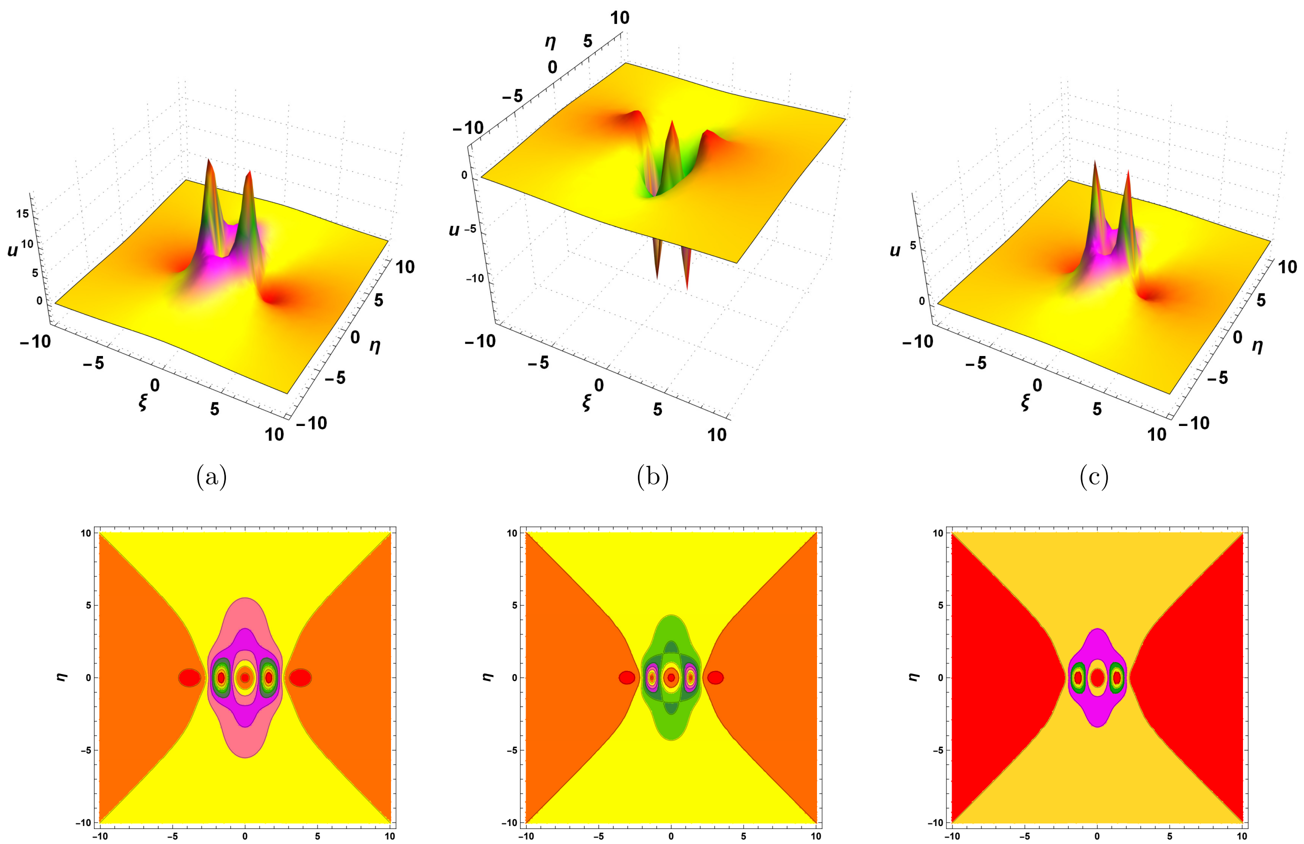


Figure 2: Dynamical profiles of (18) in the starting variables x, y, z, t under transformations $\xi = x + t$ and $\eta = y + z$. **(a)-(c)** and **(d)-(f)** depict 3D profiles in xt- and xy-planes, respectively.

$$
\begin{gathered}
s_{2,10} = \frac{(\alpha_3+1)^4 s_{10,2}}{(\alpha_4+\alpha_5)^4}, \qquad s_{4,0} = -\frac{5187875\alpha_2^4(\alpha_4+\alpha_5)s_{10,2}}{18(\alpha_3+1)^5}, \qquad s_{4,2} = -\frac{36750\alpha_2^3 s_{10,2}}{(\alpha_3+1)^3}, \\
s_{4,4} = \frac{18725\alpha_2^2 s_{10,2}}{3(\alpha_3+1)(\alpha_4+\alpha_5)}, \qquad s_{4,6} = -\frac{730\alpha_2(\alpha_3+1)s_{10,2}}{3(\alpha_4+\alpha_5)^2}, \qquad s_{4,8} = \frac{5(\alpha_3+1)^3 s_{10,2}}{2(\alpha_4+\alpha_5)^3}, \\
s_{6,0} = -\frac{37730\alpha_2^3(\alpha_4+\alpha_5)s_{10,2}}{9(\alpha_3+1)^4}, \qquad s_{6,2} = \frac{9310\alpha_2^2 s_{10,2}}{3(\alpha_3+1)^2}, \qquad s_{6,4} = -\frac{770\alpha_2 s_{10,2}}{3(\alpha_4+\alpha_5)}, \\
s_{6,6} = \frac{10(\alpha_3+1)^2 s_{10,2}}{3(\alpha_4+\alpha_5)^2}, \qquad s_{8,0} = \frac{245\alpha_2^2(\alpha_4+\alpha_5)s_{10,2}}{2(\alpha_3+1)^3}, \qquad s_{8,2} = -\frac{115\alpha_2 s_{10,2}}{\alpha_3+1}, \\
s_{8,4} = \frac{5(\alpha_3+1)s_{10,2}}{2(\alpha_4+\alpha_5)}, \qquad s_{10,0} = -\frac{49\alpha_2(\alpha_4+\alpha_5)s_{10,2}}{3(\alpha_3+1)^2}, \qquad s_{10,2} = s_{10,2}, \\
s_{12,0} = \frac{(\alpha_4+\alpha_5)s_{10,2}}{6(\alpha_3+1)}. \qquad (24)
\end{gathered}
$$

We obtain the solution by putting the equation (23) with the values (24) into (10) as

$$
u(\xi, \eta) = u_3 = \frac{12\alpha_2}{\alpha_1}(\log f_3)_{\xi\xi}. \qquad (25)
$$



(a) (b) (c)

(d) (e) (f)

Figure 3: Dynamics of 2^{nd}-order rogue waves for (22) in transformed variables ξ and η. **(d)-(f)** are the contours for **(a)-(c)** in $\xi\eta$-plane.

3 Results and Analysis

This work studied the rogue waves as an extraordinary oceanic phenomenon characterized by their immense height, steepness, and sudden appearance. Their formation shows the link to nonlinear interactions, where energy from smaller waves combines through constructive interference or the interaction between ocean currents and opposing waves. Despite their rarity, rogue waves concentrate immense power in a small area capable of causing catastrophic damage. Predicting these waves remains a significant and ongoing challenge in oceanography, though advances in wave modeling and satellite technology continue to improve our understanding of these formidable forces of nature. The investigated equation showed the rogue wave structures in transformed variables ξ and η with appropriate parameter values utilizing direct symbolic approach. The first-order rogue solution generated single rogue wave solution, and second and third-order rogue solutions gave the interactions of two and three rogue waves, respectively. The dynamics of rogue wave solutions have been shown in transformed variables ξ, η, and in the starting variables x, y, z, t in $\xi\eta$, xt, and xy planes. The analytical and dynamical findings are as follows:

- Figure-1 and 2 depicts the single rogue waves of first order having singularities at $\xi = \eta = 0$. For all three plots in Figure-1, positive and negative direction of ξ shows the bright and the dark part of the rogue wave dynamics. Depending on the constant parameters, the rogue wave shows the nature of steeped and immense height from 15 to 40 units of u in the forming local region. Figure-2 illustrates

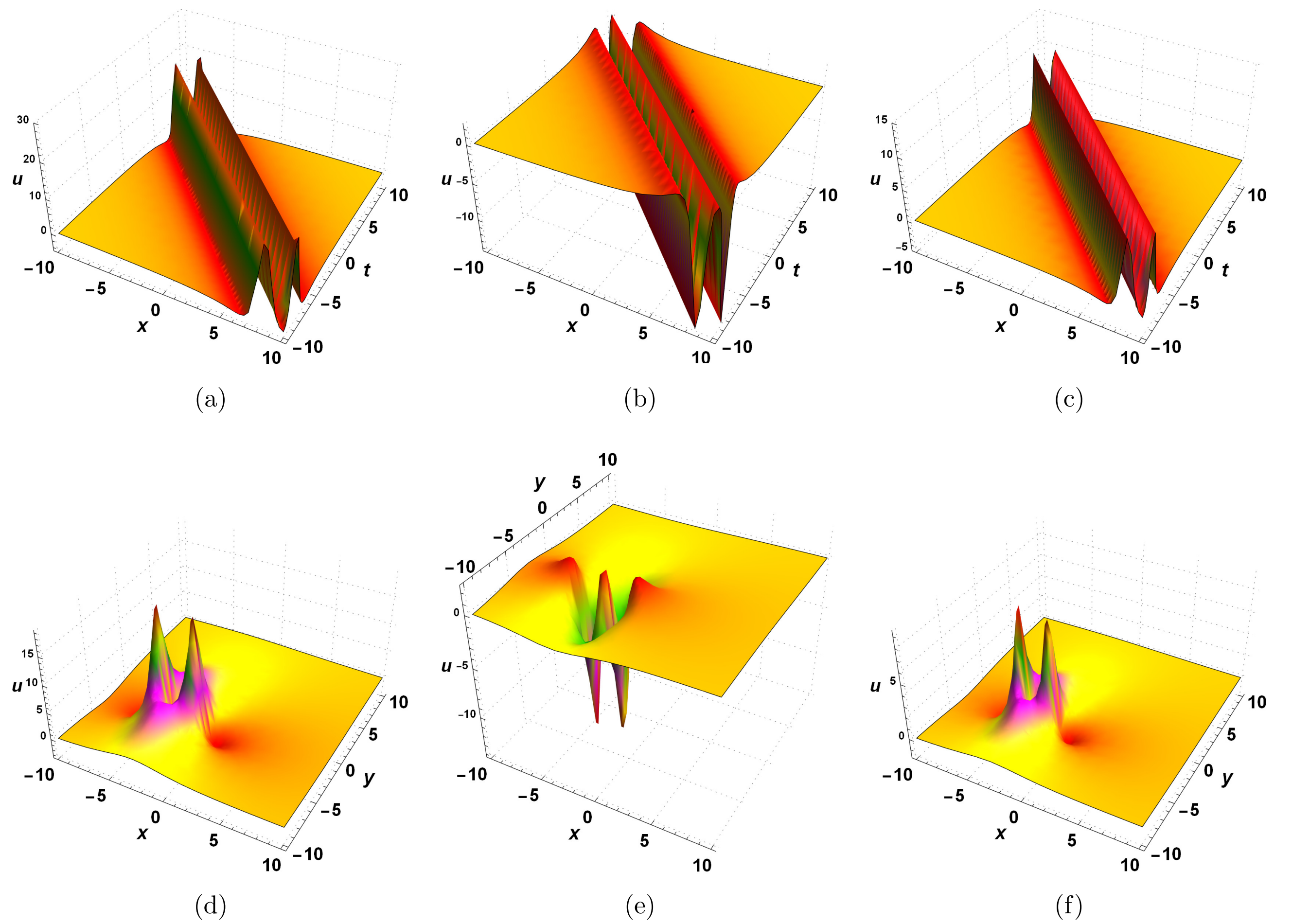


Figure 4: Dynamical profiles of (22) in the starting variables x, y, z, t under transformations $\xi = x + t$ and $\eta = y + z$. **(a)-(c)** and **(d)-(f)** depict 3D profiles in xt- and xy-planes, respectively.

the single rogue wave structures in the original variables x, y, z, t. (a)-(c) shows the soliton nature w.r.t. time variable in xt-plane with spacial coordinates as $y = z = 0$, and (d)-(f) shows the single rogue waves in xy-plane with $z = t = 3$. The showed rogue waves for both figures have the parameter values as **(a)** $\alpha_1 = -1, \alpha_2 = -1, \alpha_3 = \alpha_4 = \alpha_5 = 1$; **(b)** $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 5, \alpha_4 = 1, \alpha_5 = 1$; and **(c)** $\alpha_1 = -1, \alpha_2 = -2, \alpha_3 = 5, \alpha_4 = 1, \alpha_5 = 2$.

- In Figure-3 and 4, we illustrate the rogue waves of second order which show the interactions of two rogue waves. For all three graphs in Figure-3, the two rogue waves intersect at $\xi = \eta = 0$ with having a void area between their bight and dark wave parts which makes them dangerous to sail the ships near them. Interaction of these two rogue waves is dominating to each other to form a larger wave than the small waves in a relatively small area. Figure-4 shows the two rogue wave structures in the original variables x, y, z, t. (a)-(c) shows the solitoninc nature w.r.t. time variable, in xt-plane with $y = z = 0$, and (d)-(f) shows the two rogue waves in xy-plane with $z = t = 3$. The showed second-order rogue waves for both figures have the parameter values as **(a)** $\alpha_1 = -1, \alpha_2 = -1, \alpha_3 = \alpha_4 = \alpha_5 = 1$; **(b)** $\alpha_1 = 2, \alpha_2 = -1, \alpha_3 = 2, \alpha_4 = 1, \alpha_5 = 2$; and **(c)** $\alpha_1 = -3, \alpha_2 = -1, \alpha_4 = 1, \alpha_3 = \alpha_5 = 2$.

- Figure-5 and 6 show the 3^{rd}-order rogue waves that depict the three rogue waves having their interactions and creating a void area among their interactions with a sharp and steeped wave forms.

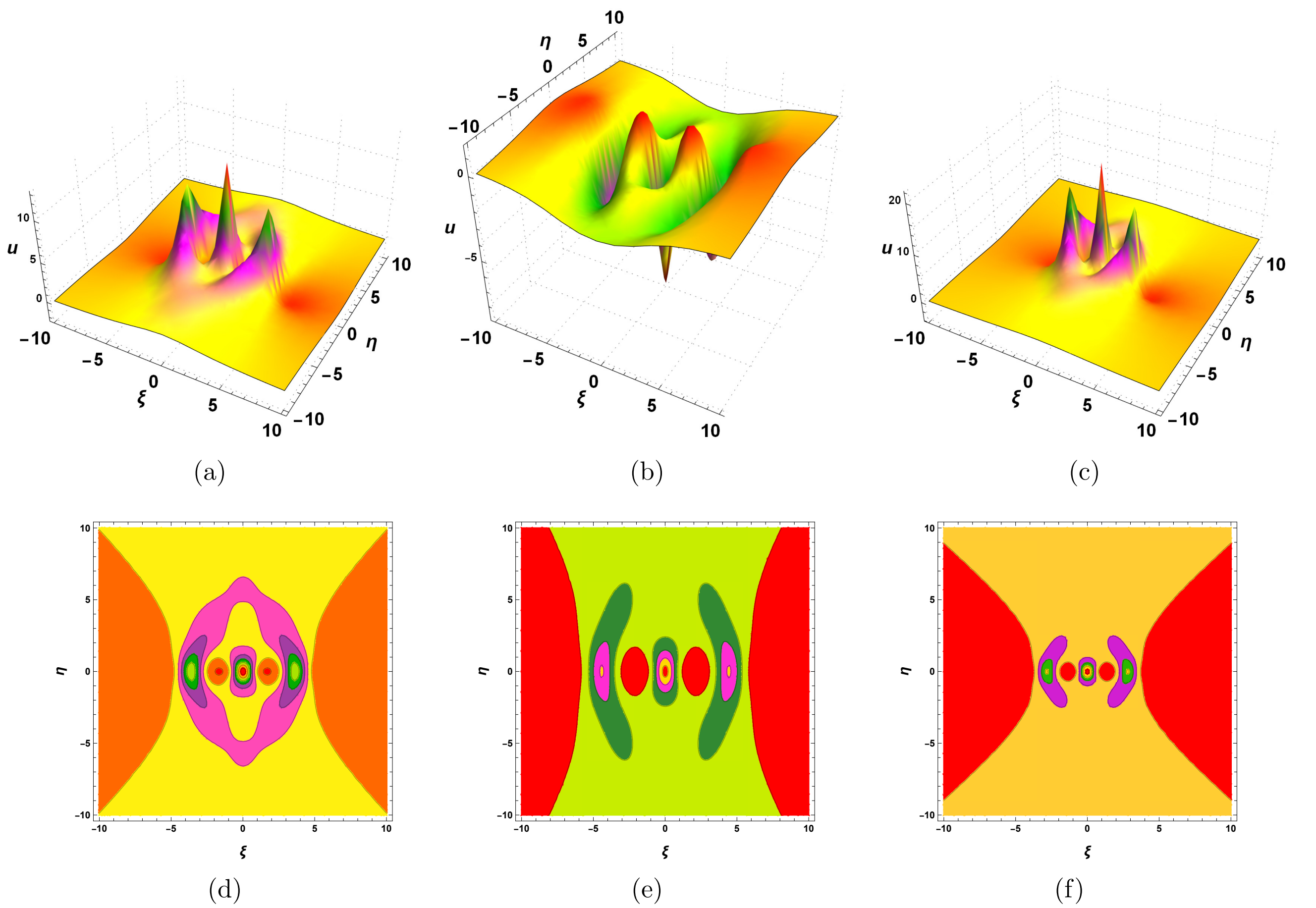


Figure 5: Dynamics of 3^{rd}-order rogue waves for (25) in transformed variables ξ and η. **(d)-(f)** are the contours for **(a)-(c)** in $\xi\eta$-plane.

Interaction of these three rogue waves is dominating to each other to form larger waves than the small waves in a relatively small area which makes them harmful than the ordinary waves. For all plots in Figure-5, the three rogue waves depict their bight and dark parts on intersections. Figure-6 shows the three rogue wave structures in the original variables x, y, z, t. (a)-(c) shows the soliton behavior with respect to time variable in xt-plane with $y = z = 0$, and (d)-(f) shows the three rogue waves in xy-plane. with xy-plane with $z = t = 3$. The showed third-order rogue waves for both figures have the parameter values as **(a)** $\alpha_1 = -3, \alpha_2 = -2, \alpha_3 = \alpha_4 = 2, \alpha_5 = 1$; **(b)** $\alpha_1 = 3, \alpha_2 = -2, \alpha_3 = \alpha_4 = 1, \alpha_5 = 3$; and **(c)** $\alpha_1 = -3, \alpha_2 = -2, \alpha_3 = 4, \alpha_4 = 1, \alpha_5 = 3$.

4 Conclusions

This study successfully derived higher-order rogue wave solutions for a (3+1)-dimensional generalized nonlinear wave equation in liquid-containing gas bubbles using its Hirota bilinear form. We obtained rogue waves up to the third order through the direct symbolic technique, revealing that second and third-order solutions generate two and three rogue waves, respectively. By applying the Cole-Hopf transformation, we transformed variables ξ and η to produce a bilinear equation, facilitating dynamic analysis using *Mathematica*. The graphical representations in the transformed and original variables illustrate the complex dynamics and interactions of rogue waves in nonlinear systems. Our findings highlight the significant presence and

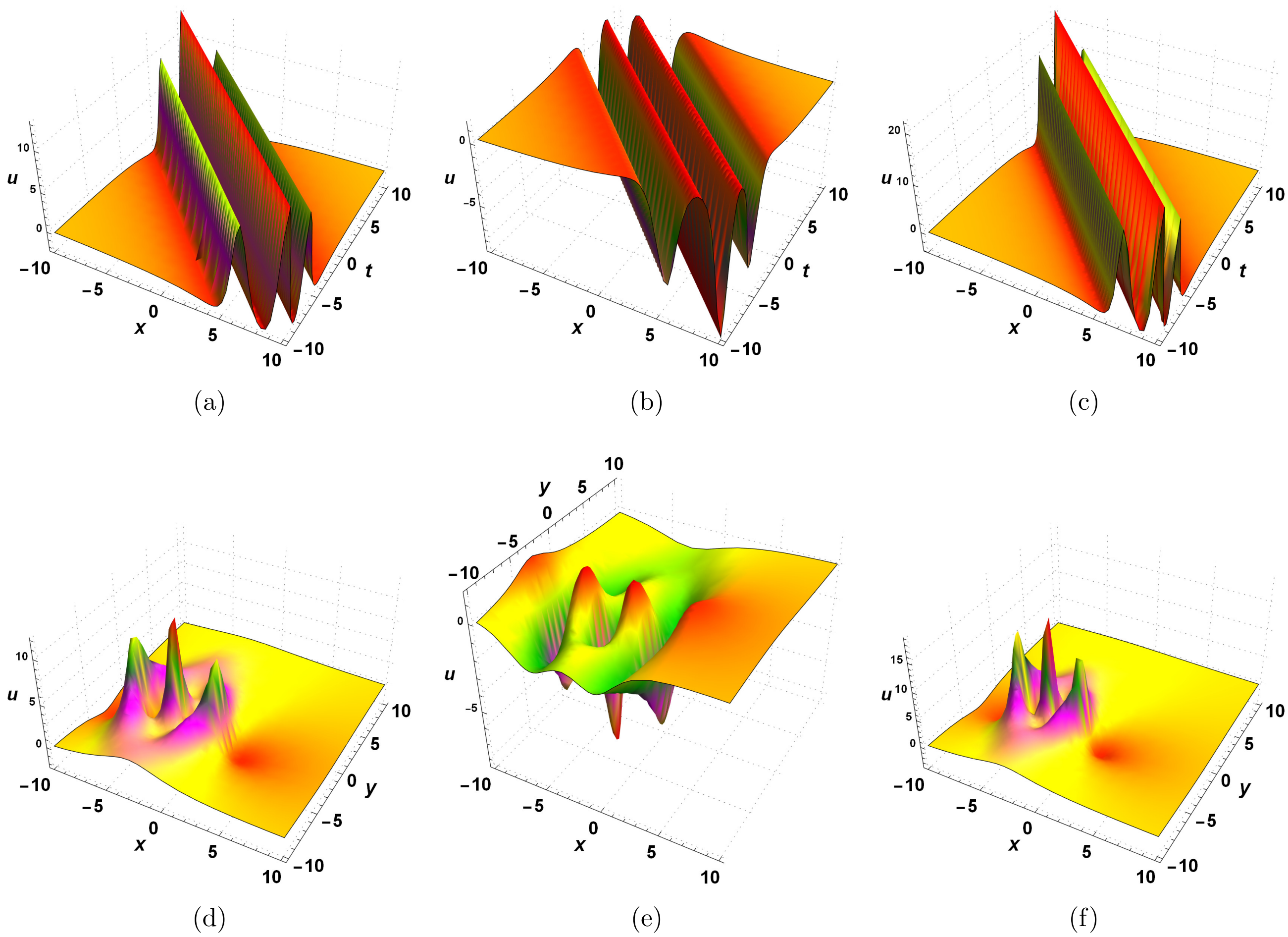


Figure 6: Dynamical profiles of (25) in the starting variables x, y, z, t under transformations $\xi = x + t$ and $\eta = y + z$. **(a)-(c)** and **(d)-(f)** depict 3D profiles in xt- and xy-planes, respectively.

intricate behavior of rogue waves, underlining their importance in understanding the evolution of large waves from smaller amplitudes. These insights are particularly relevant in nonlinear dynamics, dispersive media, and plasma physics.

The implications of this research extend across multiple scientific domains, including dusty plasma, oceanography, fiber optics, and other complex nonlinear systems. By deepening our understanding of rogue wave phenomena, this study contributes to the broader knowledge base. It paves the way for future explorations and applications in fields where critical nonlinear events are pivotal.

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Declarations

Ethics approval and consent to participate

Not applicable.

Competing interests

The authors state that there is no conflict of interest.

Authors' contributions

Both authors have agreed and given their consent for the publication of this research paper.

Data availability statement

No data is used from outer sources in this work.

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