New generalized nonlinear fifth-order KdV-type equation and its multiple soliton solutions: Hirota bilinear technique

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Abstract

We establish a generalized nonlinear fifth-order KdV-type equation using the recursion operator. This equation generalizes the Sawada-Kotera equation and the Lax equation that study the vibrations in mechanical engineering, nonlinear waves in shallow water, and other sciences. To determine the integrability, we use Painlevé analysis and construct solutions for multiple solitons by employing the Hirota bilinear technique to the established equation. It creates a bilinear form for the driven equation and utilizes the Lagrange interpolation to create a dependent variable transformation. We construct the solutions for multiple solitons and show the graphics for these built solutions. The Sawada-Kotera equation and Lax equation have various applications in mechanical engineering, plasma physics, nonlinear water waves, soliton theory, mathematical physics, and other nonlinear fields.

Keywords: Generalized KdV-type equation, Bilinear form, Painlevé analysis, Hirota technique, Logarithmic transformation, multiple solitons.

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1. Introduction

In soliton theory, two discoveries play an essential role: the direct method developed by Hirota [1] to obtain the exact solutions as a soliton, lump, breather, and rogue wave and the algorithm for Painlevé analysis [2, 3] to determine the integrability of a nonlinear partial differential equation (PDE). Investigating integrability for a PDE helps construct multiple-soliton solutions, which are exponentially localized in nature and can be reviewed by applying Painlevé analysis using symbolic systems such as *Mathematica, Matlab, Maple*, or other software.

The nonlinearity in real-world problems has attracted numerous researchers to understand a problem's dynamical behavior using nonlinear PDEs [4–9]. Xu et al. [10] and Baldwin et al. [11] gave a symbolic computation for the Painlevé test, making it more accessible for a new researcher to check the integrability of a nonlinear PDE. The Hirota technique gets the soliton solutions for a nonlinear PDE if the equation fulfills integrability. Investigators use the Hirota bilinear method in different areas such as mechanical engineering, nonlinear dynamics, mathematical physics, oceanography, soliton theory, and other sciences. Many techniques other than the Hirota bilinear method construct the exact or closed-form solutions of a nonlinear PDE. These include the Inverse scattering method [12, 13], Equivalence transformation [14], Velocity resonance method [15], Lie symmetry analysis [16, 17], simplified Hirota method [18, 19], Darboux transformation [20, 21], Bäcklund transformation [22, 23], and others.

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The nonlinear evolution equation

$$u_t + \lambda u u_x + u_{xxx} = 0, \tag{1}$$

where $u = u(x,t)$ and λ is a constant, gives the standard Korteweg-de Vries (KdV) equation [1] for $\lambda = 6$ that studies shallow-water waves, dispersive waves, and integrable systems in various domains of nonlinear sciences. The KdV equation has a recursion operator [24–26]

$$L := \partial_{xx} + 4u + 2u_x\partial_x^{-1}, \tag{2}$$

and can be represented as

$$u_t + L(u_x) = 0. \tag{3}$$

The recursion operator (2) can be used to develop the higher-order nonlinear equations for a PDE if exist. The equation (3) with operator (2) gives the fifth-order nonlinear evolution equation for the equation (1) as

$$u_t + 5\lambda u^2 u_x + (2+3\lambda)u_x u_{xx} + (4+\lambda)u u_{xxx} + u_{xxxxx} = 0, \tag{4}$$

for the parameter λ, and can be written in the standard form as

$$u_t + Au^2 u_x + Bu_x u_{xx} + Cu u_{xxx} + u_{xxxxx} = 0, \tag{5}$$

where $A = 5\lambda$, $B = 2 + 3\lambda$, and $C = 4 + \lambda$.
(i) For $\lambda = 1$, equation (5) results to the Sawada-Kotera (SK) equation [27–30] as

$$u_t + 5u^2 u_x + 5u_x u_{xx} + 5u u_{xxx} + u_{xxxxx} = 0. \tag{6}$$

(ii) For $\lambda = 6$, equation (5) results to the Lax equation [29–32] as

$$u_t + 30u^2 u_x + 20u_x u_{xx} + 10u u_{xxx} + u_{xxxxx} = 0. \tag{7}$$

Both equations are nonlinear KdV-type equations with three nonlinear terms; they share the same integrable properties and provide multiple-soliton solutions. We apply a logarithmic transformation to the Hirota bilinear technique to construct the numerous soliton solutions for these equations by generalized equation.

This research work establishes a generalized nonlinear evolution equation using the recursion operator and reviews the integrability of this developed equation with Painlevé analysis. General logarithmic transformation for the selected equation is achieved using Lagrange interpolation. Ensuring the integrability of an equation is essential before applying Hirota's direct method. We apply the Hirota bilinear technique to obtain multiple-soliton solutions up to fourth order with their dynamical structures.

The research is organized as follows: In Section 2, we determine the integrability of generalized equation using Painlevé analysis. Section 3 formulates the general transformation and constructs the bilinear form for the governed equation. In Section 4, we apply the Hirota bilinear technique to obtain multiple solitons and display the dynamical structures, and section 5 concludes the results and findings.

2. Painlevé analysis

To investigate the integrability of the nonlinear Eq. (5) by Painlevé analysis, we assume the solution as a Laurent series about a singular manifold $\xi(x,t)$ as

$$u(x,t) = \sum_{r=0}^{\infty} u_r(x,t)\xi^{r-\eta}, \tag{8}$$

where η is a positive integer, and $u_r = u_r(x,t); r = 0, 1, 2, ...;$. We get the value of η by substituting the Eq. (8) into Eq. (5), and by comparing the terms with dominance as

$$\eta = 2.$$

We find the behavior of leading order with respect to the resonance r as

$$u_0(x,t) = \begin{bmatrix} -6\xi_x^2 \\ \frac{-12\xi_x^2}{a} \end{bmatrix}; \quad r = -1, 6, \Lambda, \tag{9}$$

where Λ are resonances correspond to the constant λ, and it is determined as
(1) For Eq. (6), when $\lambda = 1$

$$u_0 = -6\xi_x^2; \quad r = -1, 2, 3, 6, 10,$$
$$u_0 = -12\xi_x^2; \quad r = -1, 5, 6, 12.$$

(2) For Eq. (7), when $\lambda = 6$

$$u_0 = -6\xi_x^2; \quad r = -1, 6, 8, 10,$$
$$u_0 = -2\xi_x^2; \quad r = -1, 2, 5, 6, 8.$$

The resonance $r = -1$ in all of the above statements occurs for singular manifold $\xi(x,t) = 0$ with its choice of irrational. The explicit expressions for $u_k; k = 1, 2, 3, ...$ exist with arbitrary functions for some k. Resonances r satisfy the compatibility condition identically. It shows that the established Eq. (5) is completely integrable depending on the parameter λ.

3. Logarithmic transformation and bilinear form

Considering the phase ψ_i in Eq. (5) as

$$\psi_i = \sigma_i x + \mu_i t, \tag{10}$$

where σ_i and μ_i are the constants and dispersions for $i = 1, 2, 3, ...$, respectively. By putting $u(x,t) = e^{\psi_i}$ in the Eq. (5) for its linear terms, we get the dispersion on solving the equation for μ_i as

$$\mu_i = -\sigma_i^5. \tag{11}$$

We assume the transformation in Eq. (5) as

$$u = K(lnV)_{xx}, \tag{12}$$

where K is a constant and $V = V(x,t)$. We get the values of K by substituting $V = 1 + e^{\psi_i}$ into the equation (5) as

$$K = 6 \quad \texttt{for} \quad \lambda = 1,$$
$$K = 2 \quad \texttt{for} \quad \lambda = 6.$$

Now, creating a set of pairs for λ with respect to K as $(\lambda, K) = \{(1,6),(6,2)\} = \{(\lambda, K_1),(\lambda, K_2)\}$, we apply Lagrange interpolation to get the general term for K with respect to the constant λ as

$$\begin{aligned} K &= \left(\frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\right) K_1 + \left(\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}\right) K_2 \\ K &= \frac{34-4\lambda}{5}. \end{aligned} \tag{13}$$

Thus, we get the transformation (12) for the Eq. (5) with Eq. (13) as

$$u(x,t) = \left(\frac{34-4\lambda}{5}\right)(lnV)_{xx}, \tag{14}$$

or

$$u(x,t) = \left(\frac{34-4\lambda}{5}\right)\left(\frac{VV_{xx}-V_x^2}{V^2}\right). \tag{15}$$

On utilizing the algorithm by Kumar-Mohan [33] for constructing the bilinear form for a class of KdV-type equations, the equation (5) with equation (15) can be transformed into the biliner form as

$$s(D_xD_t + D_x^6)V.V = 0, \tag{16}$$

where $s = 3K$ and D is the Hirota's bilinear operator [1], defined as

$$D_X^m D_T^n P(X,T)Q(X,T) = \left(\frac{\partial}{\partial_X} - \frac{\partial}{\partial_{X'}}\right)^m \left(\frac{\partial}{\partial_T} - \frac{\partial}{\partial_{T'}}\right)^n P(X,T)Q(X',T')|_{X=X',T=T'}. \tag{17}$$

For $s \neq 0$, we can write the bilinear form as

$$D_x(D_t + D_x^5)V.V = 0. \tag{18}$$

4. Formulation of multiple solitons

4.1. One soliton

To get one soliton, we consider the function V in Eq. (18) as

$$V(x,t) = 1 + e^{\psi_1} = 1 + e^{\sigma_1 x + \mu_1 t}. \tag{19}$$

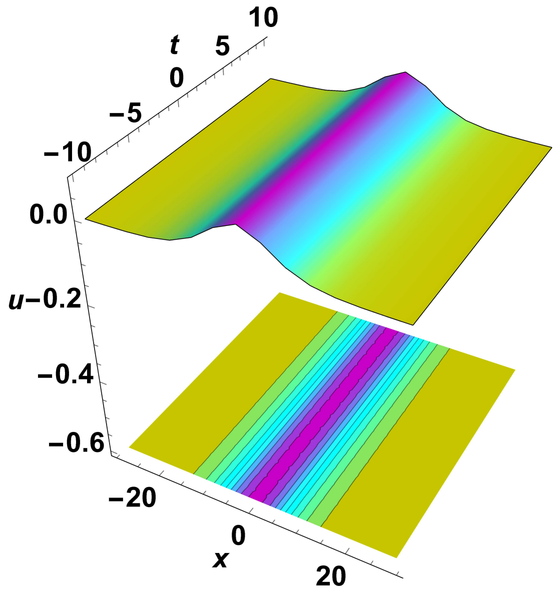
From equation (19), we can easily determine

$$V_x = \sigma_1 e^{\sigma_1 x + \mu_1 t}, \tag{20}$$

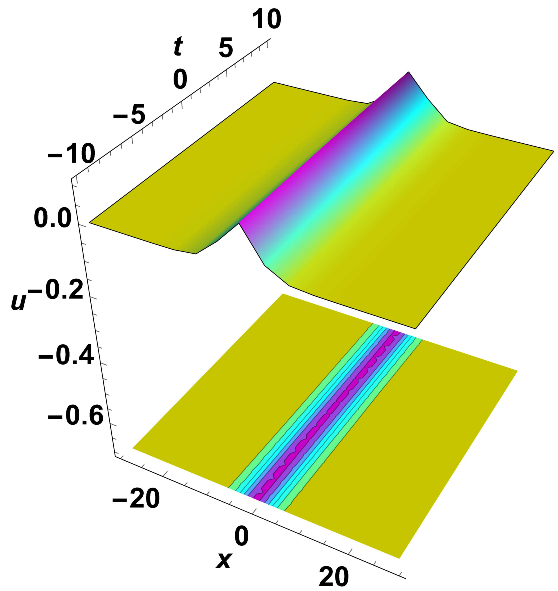
$$V_{xx} = \sigma_1^2 e^{\sigma_1 x + \mu_1 t}. \tag{21}$$

On substituting the equations (19), (20) and (21) into the equation (15), we get single-soliton solution

$$u(x,t) = \left(\frac{34-4\lambda}{5}\right)\frac{e^{x\sigma_1 + t\sigma_1^5}\sigma_1^2}{(e^{x\sigma_1} + e^{t\sigma_1^5})^2}. \tag{22}$$



(a) $\lambda = 1$ (SK equation).



(b) $\lambda = 6$ (Lax equation).

Figure 1: Single-soliton solutions with their contour plots via (15) for the function (19) with values: **(a)** $\sigma_1 = 0.25$; and **(b)** $\sigma_1 = 0.5$.

4.2. Two solitons

In the equation (5), assuming V as

$$V(x,t) = 1 + e^{\psi_1} + e^{\psi_2} + \alpha_{12} e^{\psi_1+\psi_2}, \tag{23}$$

where α_{12} is the dispersion that can be find out by putting the equations (23) and (15) into the Eq. (18). Symbolic system is used to get the values of α_{12} as

$$\alpha_{12} = \frac{(\sigma_1-\sigma_2)^2(\sigma_1^2-\sigma_1\sigma_2+\sigma_2^2)}{(\sigma_1+\sigma_2)^2(\sigma_1^2+\sigma_1\sigma_2+\sigma_2^2)} \quad \texttt{for} \quad \lambda = 1,$$

$$\alpha_{12} = \frac{(\sigma_1-\sigma_2)^2}{(\sigma_1+\sigma_2)^2} \qquad\qquad \texttt{for} \quad \lambda = 6.$$

On considering the pairs (λ, α_{12}) with respect to the parameter λ as in above expressions, we apply Lagrange interpolation to get the general term for α_{12} as

$$\alpha_{12} = \frac{(\sigma_1-\sigma_2)^2\{(5\sigma_1^2+(2\lambda-7)\sigma_1\sigma_2+5\sigma_2^2\}}{5(\sigma_1+\sigma_2)^2(\sigma_1^2+\sigma_1\sigma_2+\sigma_2^2)}. \tag{24}$$

We can envisioned the function V as follow:

$$V(x,t) = 1 + e^{\psi_i} + e^{\psi_j} + \alpha_{ij} e^{\psi_i+\psi_j}, \tag{25}$$

as

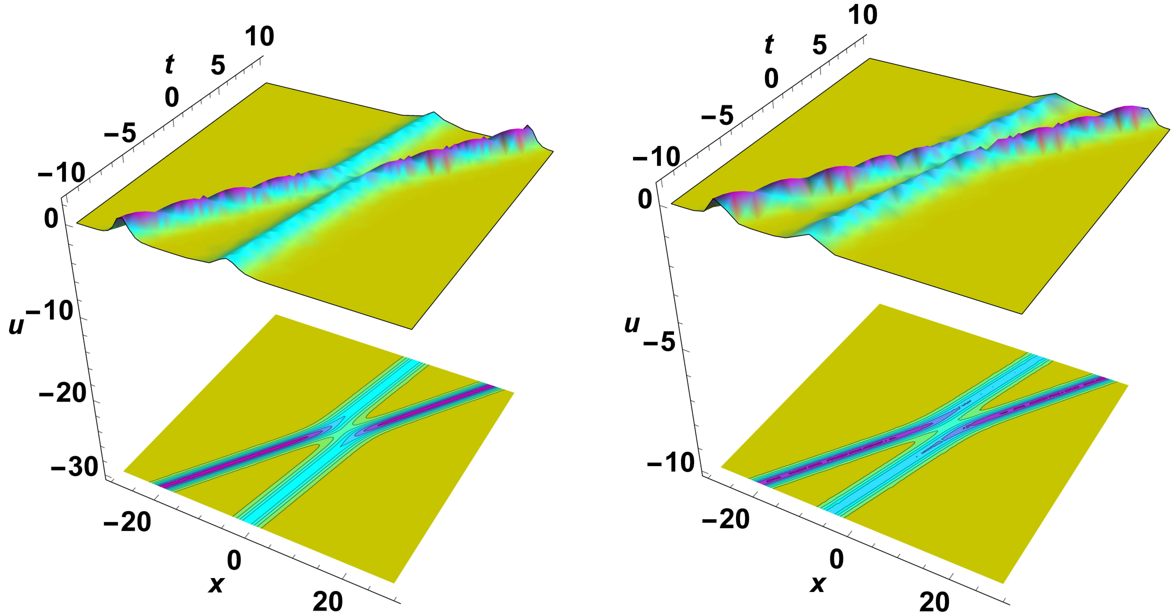
$$\alpha_{ij} = \frac{(\sigma_i-\sigma_j)^2\{(5\sigma_i^2+(2\lambda-7)\sigma_i\sigma_j+5\sigma_j^2\}}{5(\sigma_i+\sigma_j)^2(\sigma_i^2+\sigma_i\sigma_j+\sigma_j^2)}; \qquad 1 \le i < j \le M, \tag{26}$$

where M is a positive integer. Thus, by the equation (23), we can get

$$V_x = \sigma_1 e^{\psi_1} + \sigma_2 e^{\psi_1} + \alpha_{12}(\sigma_1 + \sigma_2) e^{\psi_1+\psi_2}, \tag{27}$$

$$V_{xx} = \sigma_1^2 e^{\psi_1} + \sigma_2^2 e^{\psi_1} + \alpha_{12}(\sigma_1 + \sigma_2)^2 e^{\psi_1+\psi_2}. \tag{28}$$

By puttting the Eqs (23), (27) and (28) into the Eq. (15), we determine a solution for two solitons for the equation (5).



(a) $\lambda = 1$ (SK equation). (b) $\lambda = 6$ (Lax equation).

Figure 2: Two solitons solutions with their contour plots via (15) for the function (23) with values: **(a)** $\sigma_1 = 0.9, \sigma_2 = 1.2$; and **(b)** $\sigma_1 = 1, \sigma_2 = 1.2$.

4.3. Three solitons

For three solitons, we assume V in (5) as

$$V(x,t) = 1 + e^{\psi_1} + e^{\psi_2} + e^{\psi_3} + \alpha_{12} e^{\psi_1+\psi_2} + \alpha_{13} e^{\psi_1+\psi_3} + \alpha_{23} e^{\psi_2+\psi_3} + \beta_{123} e^{\psi_1+\psi_2+\psi_3}, \tag{29}$$

where $\alpha_{ij}; 1 \leq i < j \leq 3$ can be composed from (26) and the dispersion β_{123} can be computed using symbolic computation as

$$\beta_{123} = \alpha_{12}\alpha_{13}\alpha_{23}, \tag{30}$$

which can be extrapolated for the function

$$V = 1 + e^{\psi_p} + e^{\psi_q} + e^{\psi_r} + \alpha_{pq} e^{\psi_p+\psi_q} + \alpha_{pr} e^{\psi_p+\psi_r} + \alpha_{qr} e^{\psi_q+\psi_r} + \beta_{pqr} e^{\psi_p+\psi_q+\psi_r}, \tag{31}$$

where the dispersion β_{pqr} upholds

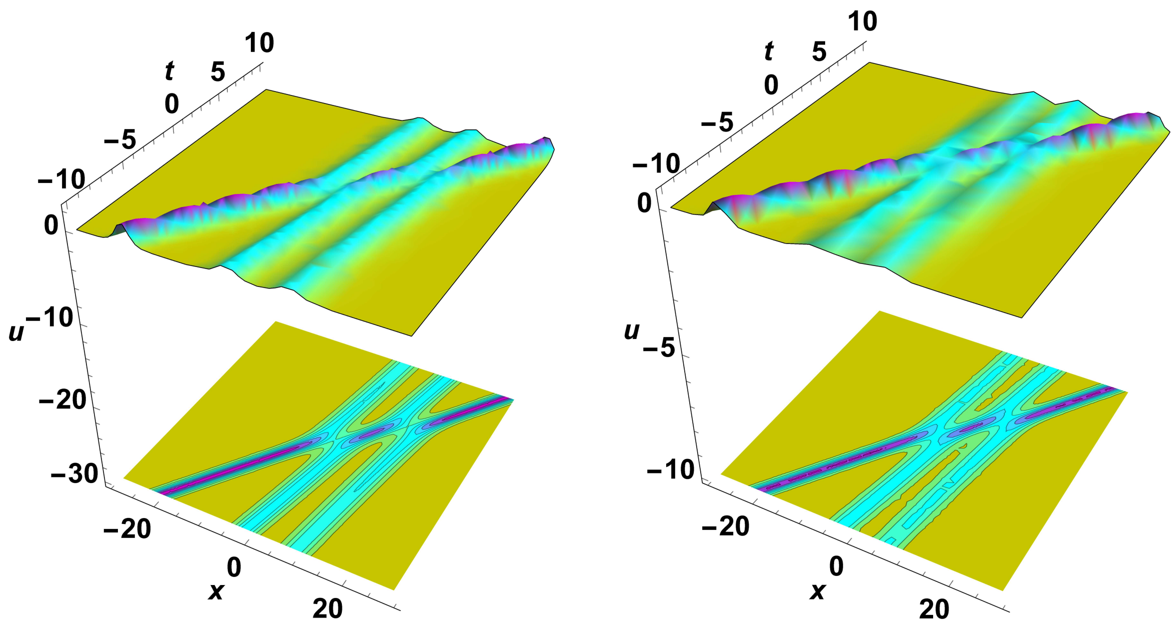
$$\beta_{pqr} = \alpha_{pq}\alpha_{pr}\alpha_{qr}; \qquad 1 \leq p < q < r \leq M, \tag{32}$$

where M is a positive integer and $\alpha_{ij}; 1 \leq i < j \leq M$ is deduced by (26). Thus, by equation (29), we have

$$V_x = \sum_{m=1}^{3} \sigma_m e^{\psi_m} + \sum_{1 \leq m < n \leq 3} (\sigma_m + \sigma_n)\alpha_{mn} e^{\psi_m + \psi_n} + \left(\sum_{m=1}^{3} \sigma_m\right) \beta_{123} e^{\psi_1 + \psi_2 + \psi_3}, \tag{33}$$

$$V_{xx} = \sum_{m=1}^{3} \sigma_m^2 e^{\psi_m} + \sum_{1 \leq m < n \leq 3} (\sigma_m + \sigma_n)^2 \alpha_{mn} e^{\psi_m + \psi_n} + \left(\sum_{m=1}^{3} \sigma_m\right)^2 \beta_{123} e^{\psi_1 + \psi_2 + \psi_3}. \tag{34}$$

By putting the equations (29), (33) and (34) into (15), we get the three-soliton solution.



(a) $\lambda = 1$ (SK equation).

(b) $\lambda = 6$ (Lax equation).

Figure 3: Three-soliton solutions with their contour plots via (15) for the function (29) with values: **(a)** $\sigma_1 = 0.9, \sigma_2 = 1.2, \sigma_3 = 0.8$; and **(b)** $\sigma_1 = 0.8, \sigma_2 = 1.2, \sigma_3 = 0.9$.

4.4. Four solitons

To determine a solution for four solitons, we consider the function V in Eq. (5) as

$$V(x,t) = 1 + \sum_{a=1}^{4} e^{\psi_a} + \sum_{1 \leq a < b \leq 4} \alpha_{ab} e^{\psi_a + \psi_b} + \sum_{1 \leq a < b < c \leq 4} \beta_{abc} e^{\psi_a + \psi_b + \psi_c} + \gamma_{1234} e^{\sum_{i=1}^{4} e^{\psi_i}}, \tag{35}$$

where $\alpha_{ab}; 1 \leq a < b \leq 4$ and $\beta_{abc}; 1 \leq a < b < c \leq 4$ conform the equations (26) and (32), respectively. Using symbolic computation, we can obtain the value of γ_{1234} as

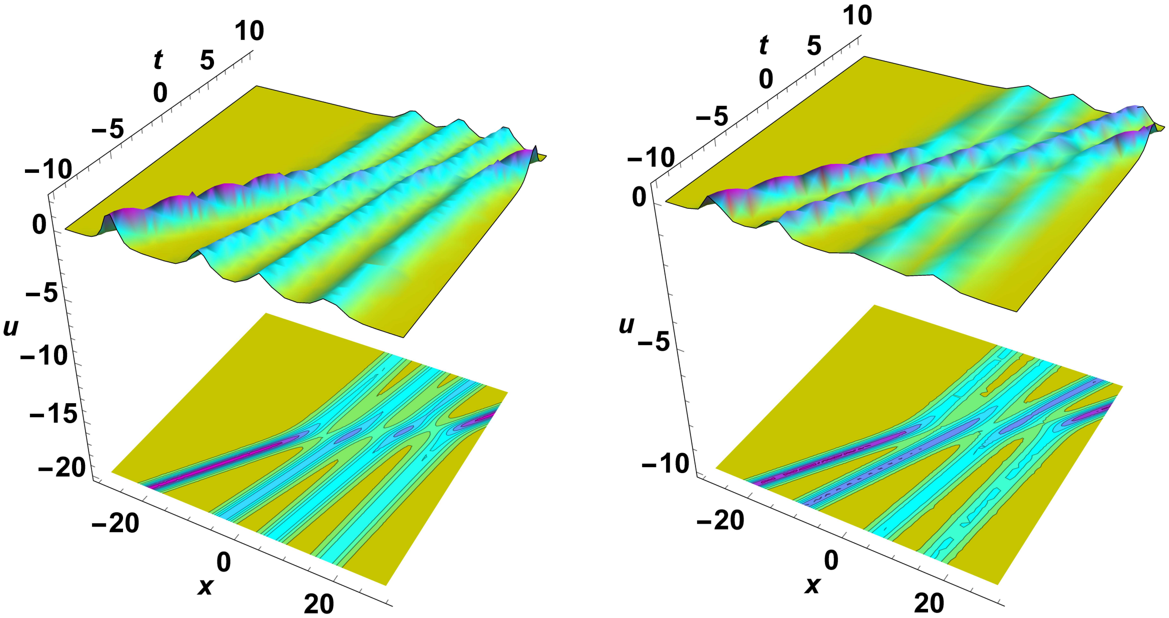
$$\gamma_{1234} = \alpha_{12}\alpha_{13}\alpha_{14}\alpha_{23}\alpha_{24}\alpha_{34}. \tag{36}$$

We deduce from (35) the following

$$V_x = \sum_{a=1}^{4} \sigma_a e^{\psi_a} + \sum_{1 \leq a < b \leq 4} (\sigma_a + \sigma_b)\alpha_{ab} e^{\psi_a + \psi_b} + \sum_{1 \leq a < b < c \leq 4} (\sigma_a + \sigma_b + \sigma_c)\beta_{abc} e^{\psi_a + \psi_b + \psi_c} + \left(\sum_{a=1}^{4} \sigma_a\right) \gamma_{1234} e^{\sum_{i=1}^{4} e^{\psi_i}}, \tag{37}$$

$$V_{xx} = \sum_{a=1}^{4} \sigma_a^2 e^{\psi_a} + \sum_{1\le a<b\le 4} (\sigma_a+\sigma_b)^2 \alpha_{ab} e^{\psi_a+\psi_b} + \sum_{1\le a<b<c\le 4} (\sigma_a+\sigma_b+\sigma_c)^2 \beta_{abc} e^{\psi_a+\psi_b+\psi_c} + \left(\sum_{a=1}^{4} \sigma_a\right)^2 \gamma_{1234} e^{\sum_{i=1}^{4} e^{\psi_i}}. \tag{38}$$

By putting the equations (35), (37) and (38) in the Eq. (15), we get the four solitons solution.



(a) $\lambda = 1$ (SK equation). (b) $\lambda = 6$ (Lax equation).

Figure 4: Four-soliton solutions with their contour plots via (15) for the function (35) with values: **(a)** $\sigma_1 = 1, \sigma_2 = 1.2, \sigma_3 = 0.8, \sigma_4 = 0.9$; and **(b)** $\sigma_1 = 0.8, \sigma_2 = 0.9, \sigma_3 = 1.2, \sigma_4 = 1.1$.

5. Conclusions

This research established a generalized nonlinear fifth-order KdV-type equation using a recursion operator. The Sawada-Kotera equation and Lax equation were generalized from the driven equation. These equations have many applications in plasma physics to study ion-acoustic waves, soliton theory to understand nonlinear waves, mechanical engineering to investigate vibrations, and other sciences. We determined the integrability of the produced equation by Painlevé analysis and constructed the solutions for multiple solitons by the Hirota bilinear method with its bilinear form. We used Lagrange interpolation to build a general transformation. We generated soliton solutions for one-to-four solitons by applying the Hirota technique and showcased dynamics for these solutions graphically. With the generalization of the SK equation and Lax equation, the established equation has many applications in nonlinear dynamics, soliton theory, oceanography, mechanical engineering, plasma physics, and other fields.

Conflict of Interest

The authors declare that they have no conflict of interest.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

[1] Hirota R. The direct method in soliton theory. Cambridge University Press; 2004.

[2] Weiss J, Tabor M, Carnevale G. The Painlevé property of partial differential equations. J Math Phys A 1983;24:522526.

[3] Weiss J. The Painlevé property for partial differential equations II: Bcklund transformation, Lax pairs, and the Schwarzian derivative. J Math Phys 1983;24:140513.

[4] Gao XY, Guo YJ, Shan WR. Symbolic computation on the long gravity water waves: scaling transformations, bilinear forms, N-soliton solutions and auto-Bäcklund transformation for a variable-coefficient variant Boussinesq system. Chaos Soliton Fract 2021;152:111392.

[5] Hu R. Diversity of Interaction Solutions to the (2+1)-Dimensional Sawada-Kotera Equation. Journal of Applied Mathematics and Physics 2018;6:1692-1703.

[6] Yang J, Fang MS, Luo L, Ma LY. From a generalized discrete NLS equation in discrete alpha helical proteins to the fourth-order NLS equation. Chaos Soliton Fract 2021;1532:111600.

[7] Hirota R, Satsuma J. N-soliton solutions of model equations for shallow water waves. J Phys Soc Jpn 1976;40(2):611612.

[8] Kumar S, Mohan B. A study of multi-soliton solutions, breather, lumps, and their interactions for Kadomtsev-Petviashvili equation with variable time coefficient using Hirota method. Phys Scr 2021;96(12):125255.

[9] Wu QL, Zhang HQ, Hang C. Breather, solitonbreather interaction and double-pole solutions of the fifth-order modified KdV equation. Applied Mathematics Letters 2021;120:107256.

[10] Xu GQ, Li ZB. Symbolic computation of the Painlevé test for nonlinear partial differential equations using Maple. Computer Physics Communications 2004;161:6575.

[11] Baldwin D, Hereman W. Symbolic Software for the Painlevé Test of Nonlinear Ordinary and Partial Differential Equations. Journal of Nonlinear Mathematical Physics 2006;13(1):90-110.

[12] Lin R, Zeng Y, Ma WX. Solving the KdV hierarchy with self-consistent sources by inverse scattering method. Physica A: Statistical Mechanics and its Applications 2001;291:287-298.

[13] Vakhnenko VO, Parkes EJ. The singular solutions of a nonlinear evolution equation taking continuous part of the spectral data into account in inverse scattering method. Chaos Soliton Fract 2012;45(6):846-852.

[14] Yong X, Huang Y, Zhao Y. Equivalence transformations of a generalized fifth-order KdV equation with variable coefficients. Partial Differential Equations in Applied Mathematics 2022;5:100224.

[15] Ma H, Huang H, Deng A. Soliton molecules, asymmetric solitons and hybrid solutions for KdVCDG equation, Partial Differential Equations in Applied Mathematics 2022;5:100214.

[16] Ali MR, Ma WX, Sadat R. Lie symmetry analysis and invariant solutions for (2+1) dimensional Bogoyavlensky-Konopelchenko equation with variable-coefficient in wave propagation. Journal of Ocean Engineering and Science 2021; https://doi.org/10.1016/j.joes.2021.08.006.

[17] Shoukry EG, Kumar S, Niwas M. Construction of multiple new analytical soliton solutions and various dynamical behaviors to the nonlinear convection-diffusion-reaction equation with power-law nonlinearity and density-dependent diffusion via Lie symmetry approach together with a couple of integration approaches. Journal of Ocean Engineering and Science 2022; https://doi.org/10.1016/j.joes.2022.01.006.

[18] Wazwaz AM. The simplified Hirota's method for studying three extended higher-order KdV-type equations. Journal of Ocean Engineering and Science 2016;1(3):181-185.

[19] Ismael HF, Murad MAS, Bulut H. Various exact wave solutions for KdV equation with time-variable coefficients. Journal of Ocean Engineering and Science 2021; https://doi.org/10.1016/j.joes.2021.09.014.

[20] Yang Y, Dong H, Chen Y. Darboux-Bäcklund transformation and localized excitation on the periodic wave background for the nonlinear Schrdinger equation. Wave Motion 2021;106:102787.

[21] Wen XY, Gao YT, Wang L. Darboux transformation and explicit solutions for the integrable sixth-order KdV equation for nonlinear waves. Applied Mathematics and Computation 2011;218:55-60.

[22] Carillo S. KdV-type equations linked via Bäcklund transformations: Remarks and perspectives. Applied Numerical Mathematics. 2019;141:81-90.

[23] Zang L, Liu QP. A super KdV equation of Kupershmidt: Bäcklund transformation, Lax pair and related discrete system. Physics Letters A 2022;422:127794.

[24] Baldwin D, Hereman W. A symbolic algorithm for computing recursion operators of nonlinear partial differential equations. Int J Comput Math 2010;87(5):10941119.

[25] Wazwaz AM. Repeated application of the recursion operator for a new hierarchy of negative-order integrable KdV equations. Waves in Random and Complex Media 2018;30(2):300-307.

[26] Olver PJ. Evolution equations possessing infinitely many symmetries. J Math Phys 1977;18(6):12121215.

[27] Zhang Y. Lie symmetry analysis and exact solutions of the SawadaKotera equation. Turk J Math 2017;41:158167.

[28] Bilige S, Chaolu T. An extended simplest equation method and its application to several forms of the fifth-order KdV equation. Applied Mathematics and Computation 2010;216(11):3146-3153.

[29] Kumar S, Mohan B. Generalized fifth-order nonlinear evolution equation for the Sawada-Kotera, Lax, and Caudrey-Dodd-Gibbon equations in plasma physics: Painlevé analysis and multi-soliton solutions, Physica scripta 2022;97(3):035201

[30] Wang G, Kara AH, Fakhar K, Jose VG, Biswas A. Group analysis, exact solutions and conservation laws of a generalized fifth order KdV equation. Chaos Soliton Fract 2016;86:8-15.

[31] Wazwaz AM. N-soliton solutions for the combined KdV-CDG equation and the KdV-Lax equation. Applied Mathematics and Computation 2008;203:402-407.

[32] Wazwaz AM. Two-mode fifth-order KdV equations: necessary conditions for multiple-soliton solutions to exist. Nonlinear Dyn 2017;87:1685-1691.

[33] Kumar S, Mohan B. A novel and efficient method for obtaining Hirota's bilinear form for the nonlinear evolution equation in (n+1) dimensions, Partial Differential Equations in Applied Mathematics 2022;5:100274.